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Approximate Polymorphisms

Joint work with
Noam Lifshitz, Dor Minzer, Elchanan Mossel

Yuval Filmus, 7 March 2021, TAU Combinatorics Seminar

Judgement Aggregation

Defendant is guilty if they have the **means** and the **motive**

	Means?	**Motive?**	**Guilty?**
Ehud	Yes	No	No
Shamgar	No	Yes	No
Deborah	Yes	Yes	Yes
Majority	**Yes**	**Yes**	**No**

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	Means?	**Motive?**	**Guilty?**
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Shamgar	No	Yes	No
Deborah	Yes	Yes	Yes
Majority	**Yes**	**Yes**	**No**

Inconsistent!

Polymorphisms

Polymorphisms

Given:

Polymorphisms

Given:

Predicate $P \subseteq \{0,1\}^k$

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Function f: $\{0,1\}^n \to \{0,1\}$

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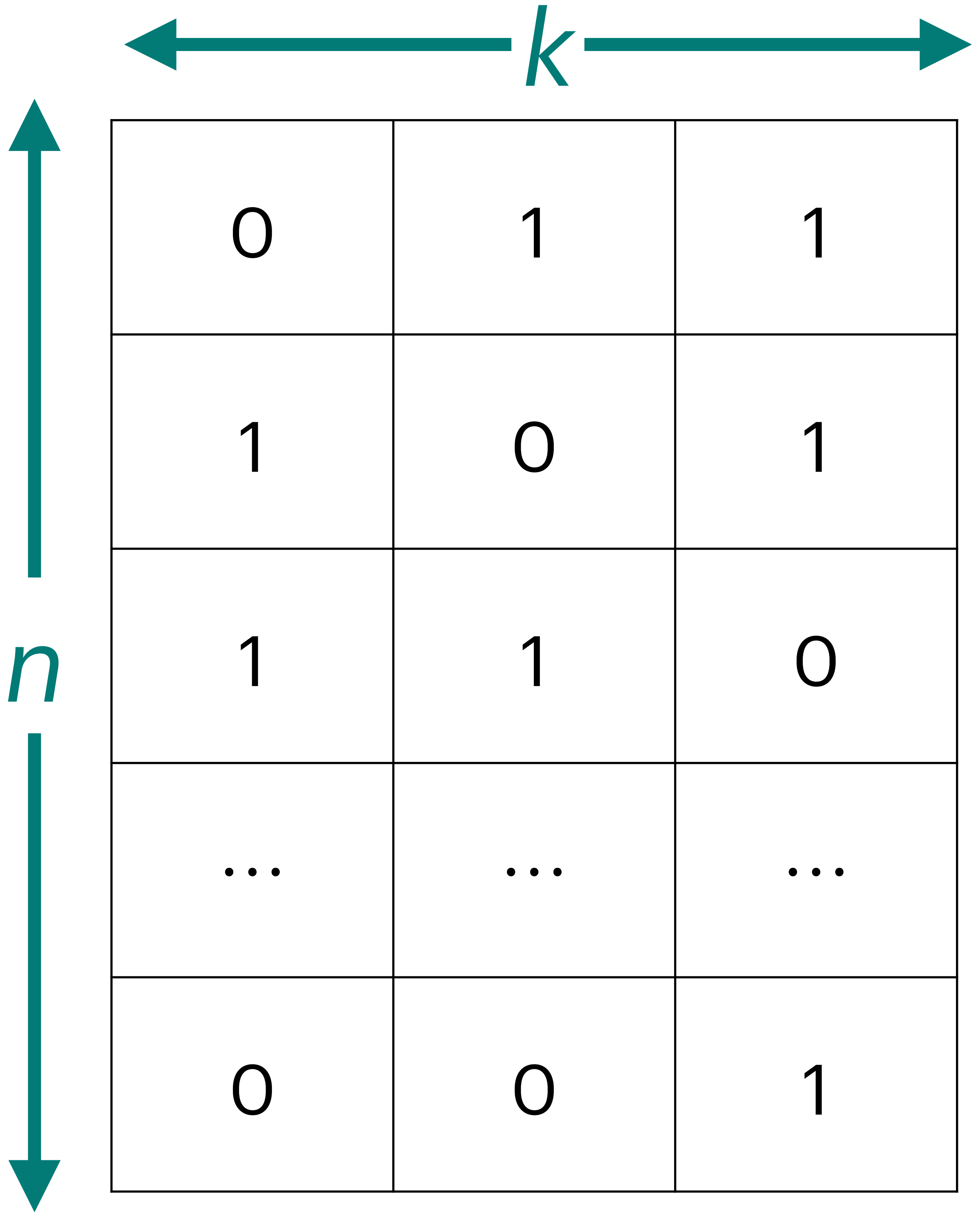

0	1	1
1	0	1
1	1	0
…	…	…
0	0	1

Polymorphisms

Given:

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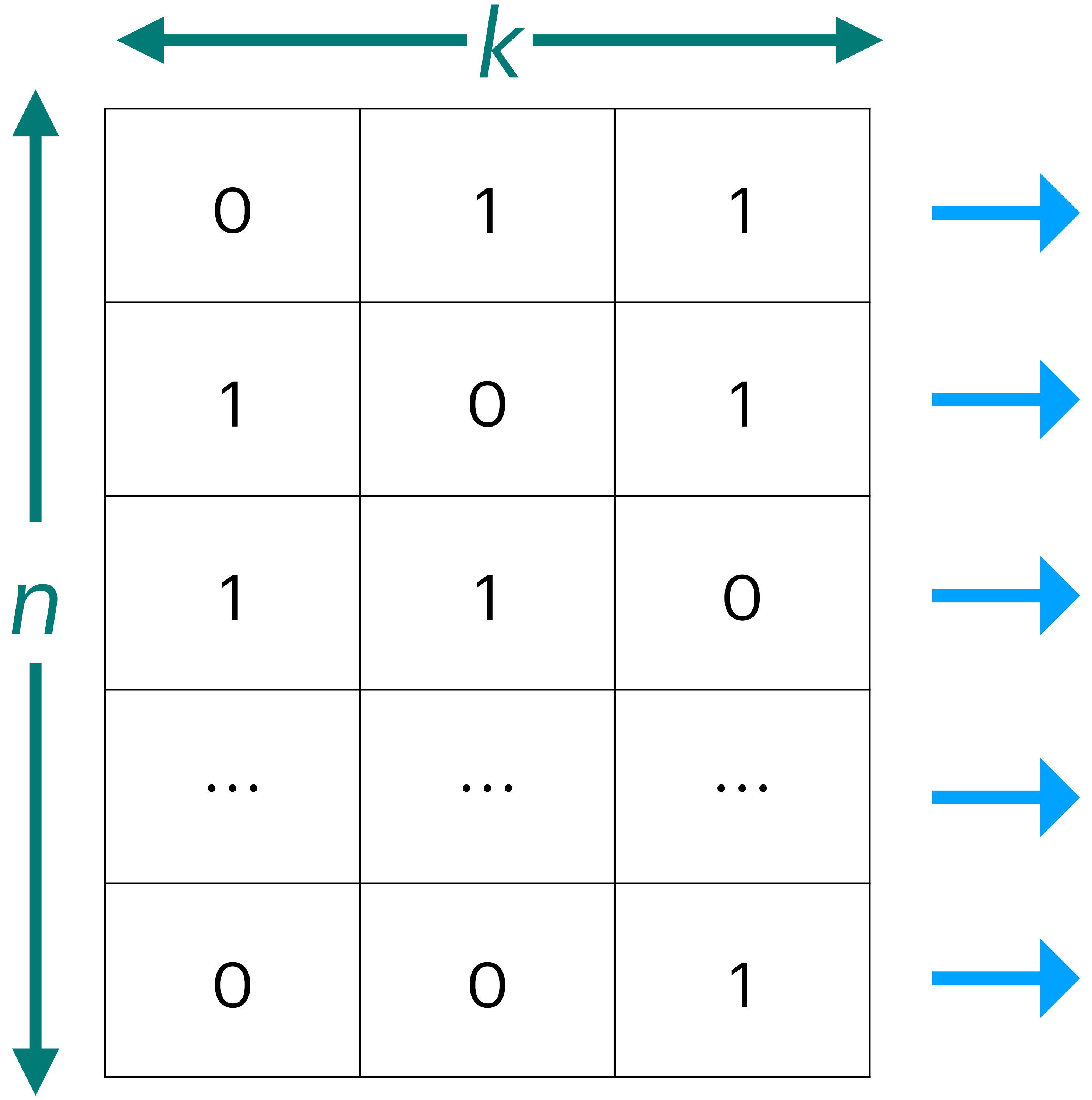
Function f: $\{0,1\}^n \rightarrow \{0,1\}$



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All rows satisfy property P

Polymorphisms

Given:

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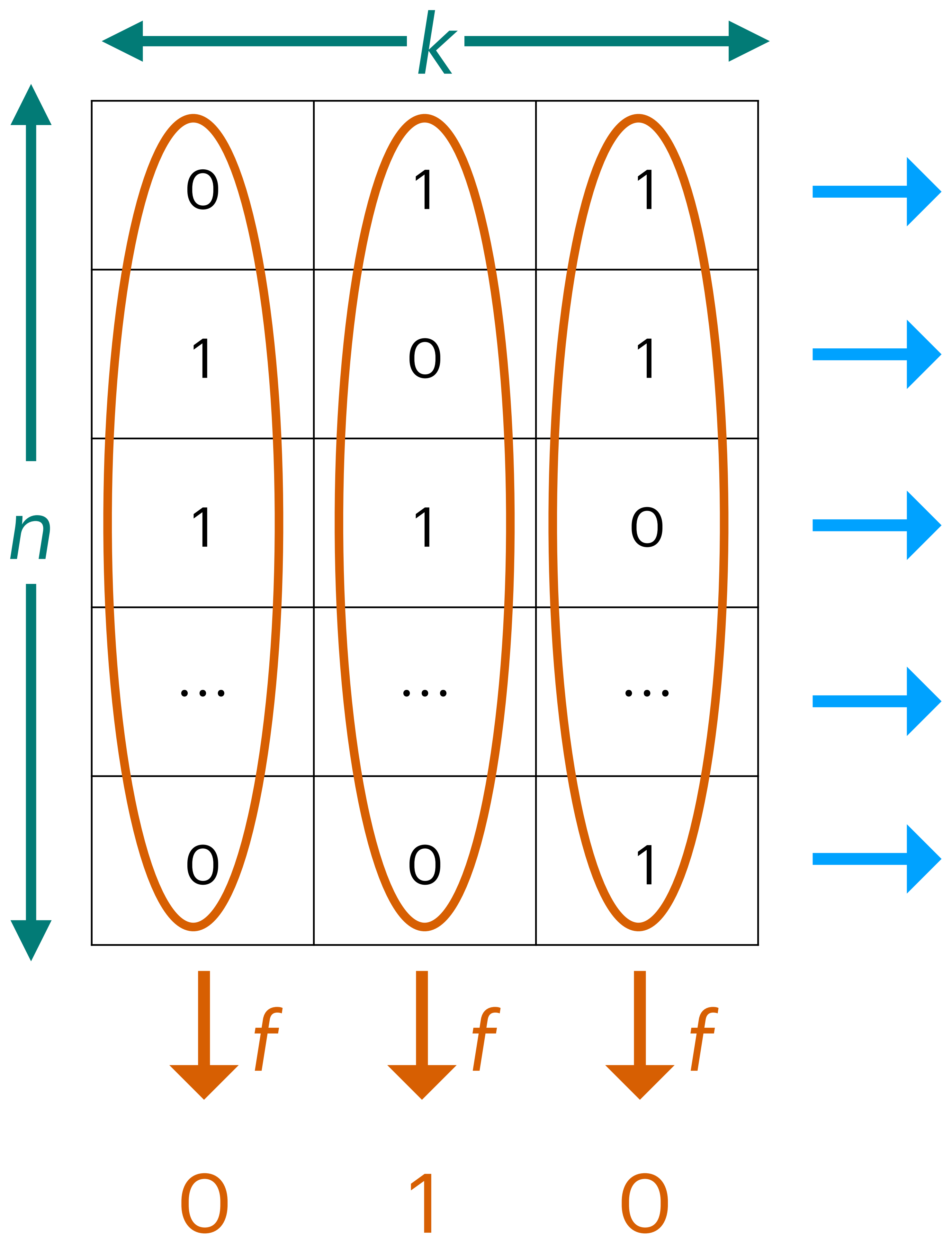

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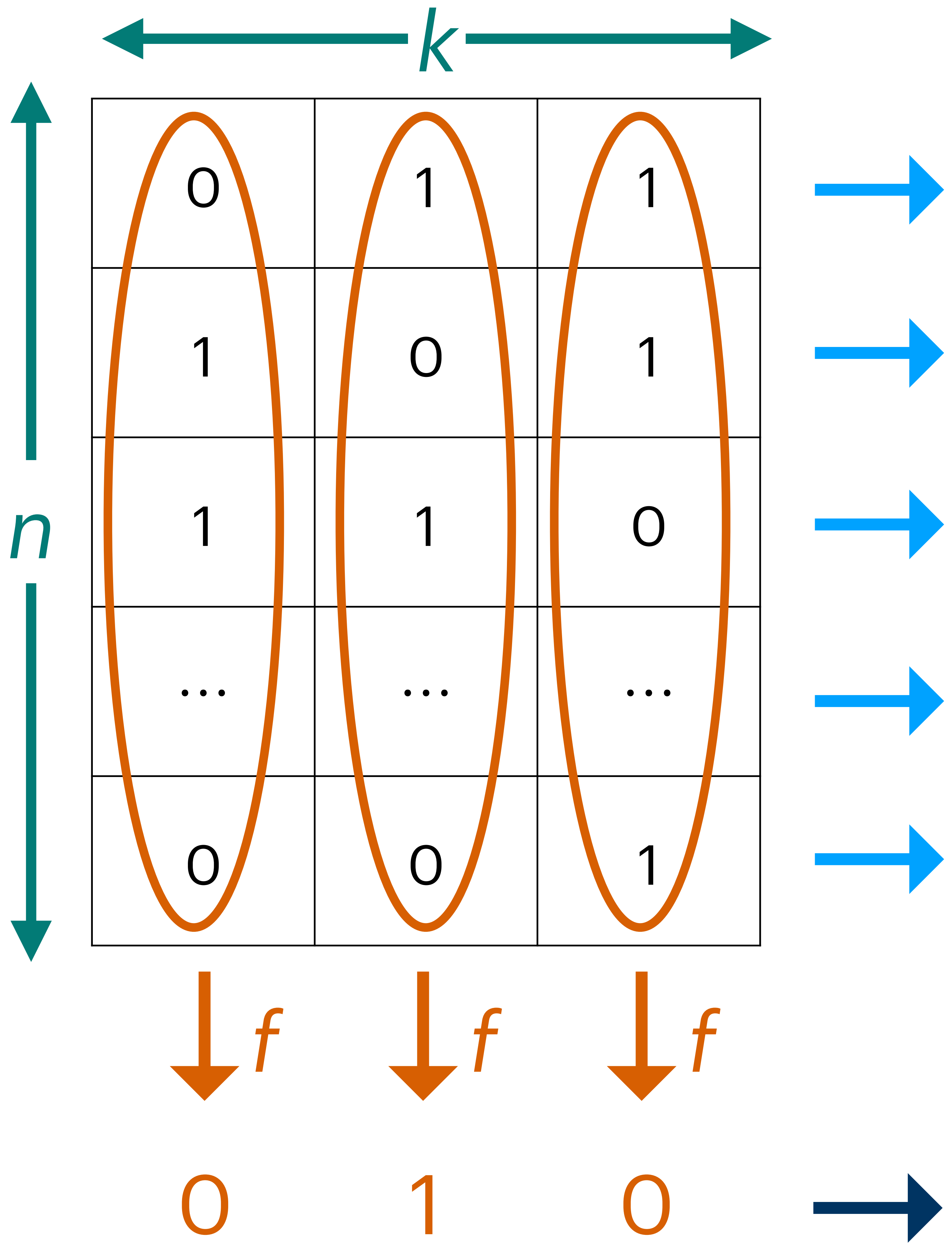

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1	0	1
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…	…	…
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0 1 0

All rows satisfy property P

Should satisfy P

Polymorphisms

Given:

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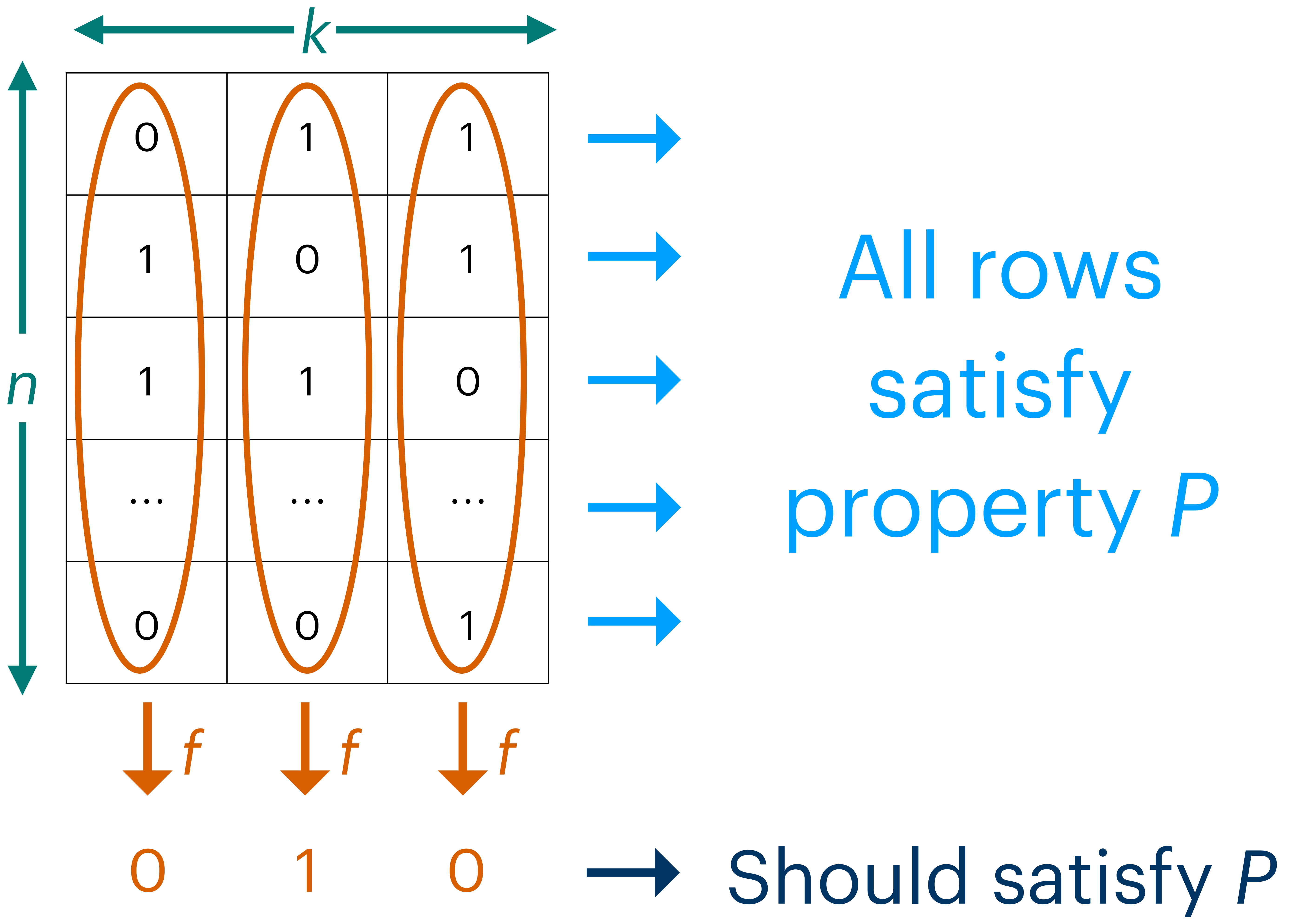
Function f: $\{0,1\}^n \rightarrow \{0,1\}$

"f is a polymorphism of P"



0	1	1
1	0	1
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…	…	…
0	0	1

All rows satisfy property P

0 1 0 → Should satisfy P

Example: Not-All-Equal

Voters

A>B?	B>C?	C>A?
0	1	1
1	0	1
1	1	0
…	…	…
0	0	1

Example: Not-All-Equal

Voters

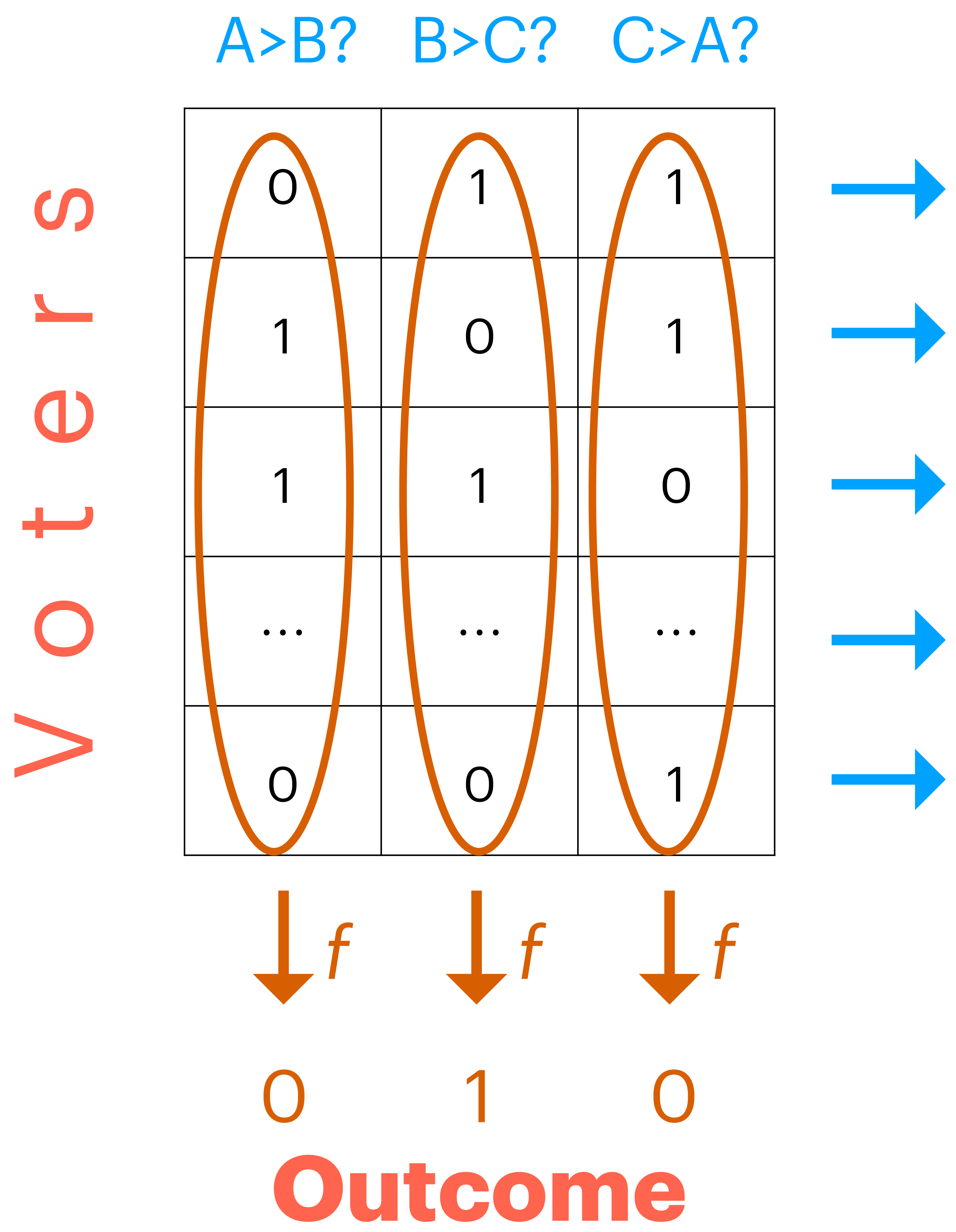
A>B?	B>C?	C>A?
0	1	1
1	0	1
1	1	0
…	…	…
0	0	1

Forbidden rows:

0 0 0

1 1 1

Example: Not-All-Equal



Voters

A>B?	B>C?	C>A?
0	1	1
1	0	1
1	1	0
…	…	…
0	0	1

$\downarrow f$ $\downarrow f$ $\downarrow f$

0 1 0

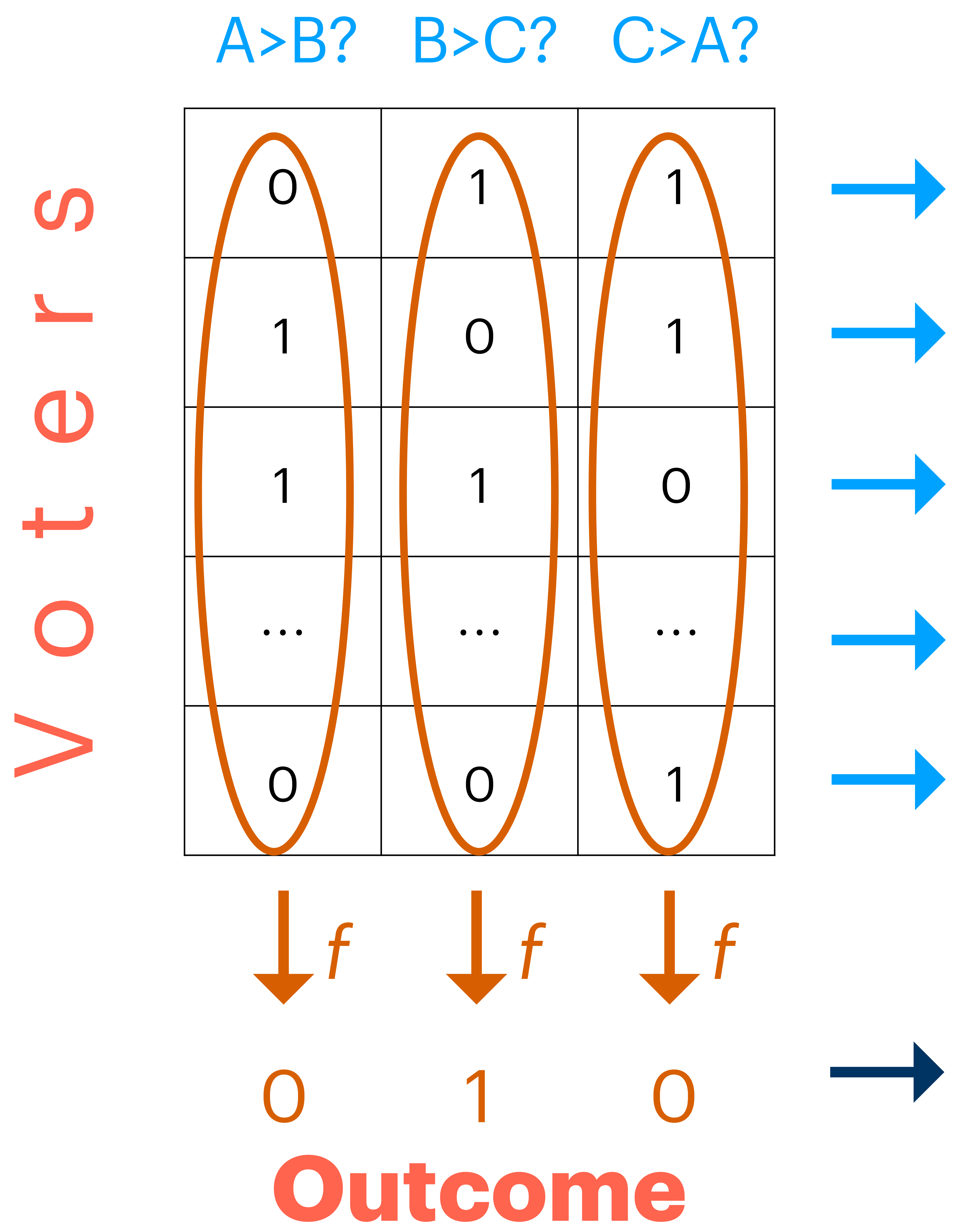
Outcome

Forbidden rows:

0 0 0

1 1 1

Example: Not-All-Equal



Voters

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1	0	1
1	1	0
…	…	…
0	0	1
0	1	0

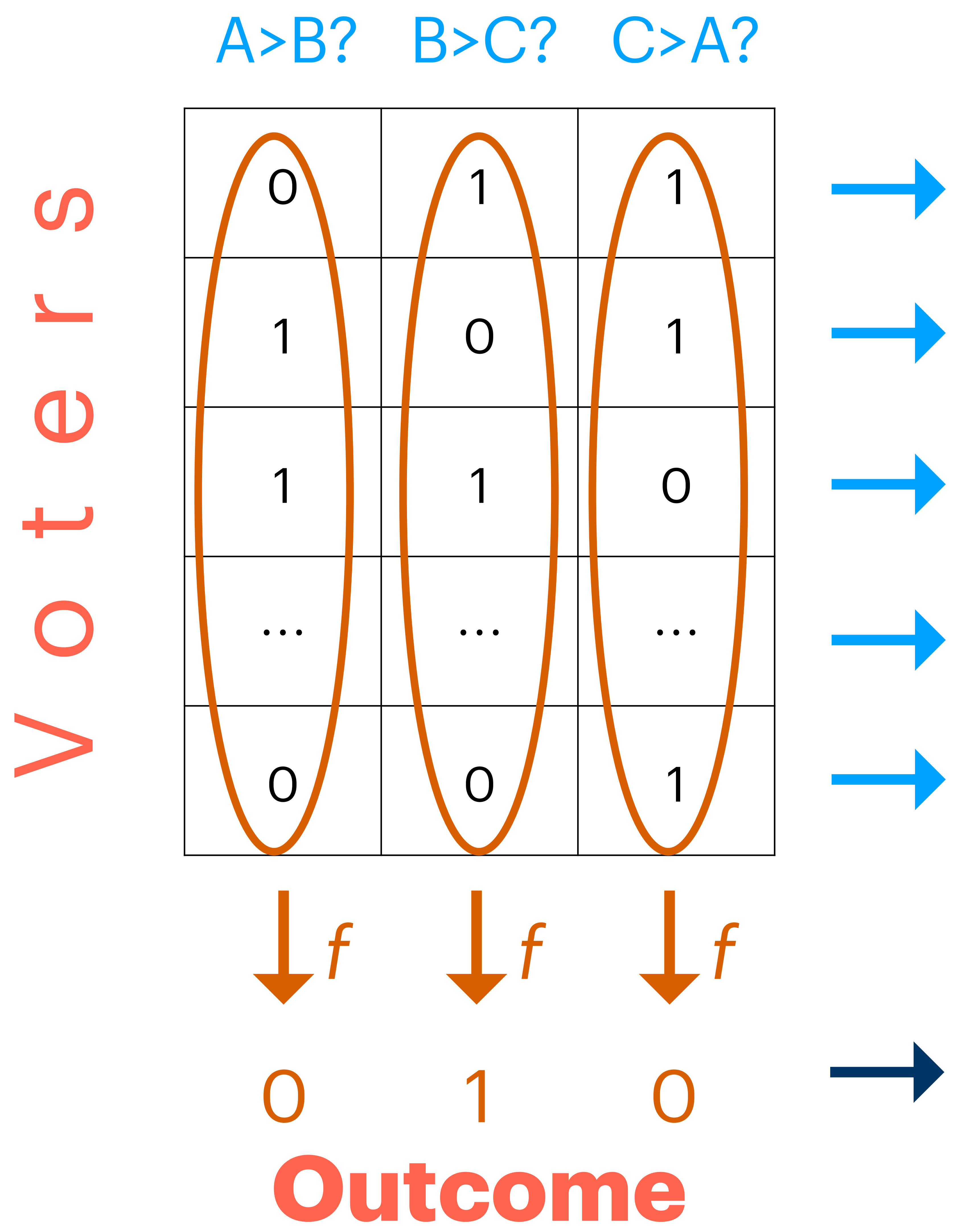
Outcome

Forbidden rows:
0 0 0
1 1 1

Outcome must be legal

Example: Not-All-Equal

Polymorphisms:
Dictators (*i*-th row)
(Arrow’s theorem)



A>B?	B>C?	C>A?
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1	1	0
…	…	…
0	0	1

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Outcome

Forbidden rows:
0 0 0
1 1 1

Outcome must be legal

Example: Even Parity

A function $f\colon \{0,1\}^n \to \{0,1\}$ is *linear* if $f(x \oplus y) = f(x) \oplus f(y)$

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1	0	1
1	1	0
…	…	…
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1	1	0
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→ Rows have even parity

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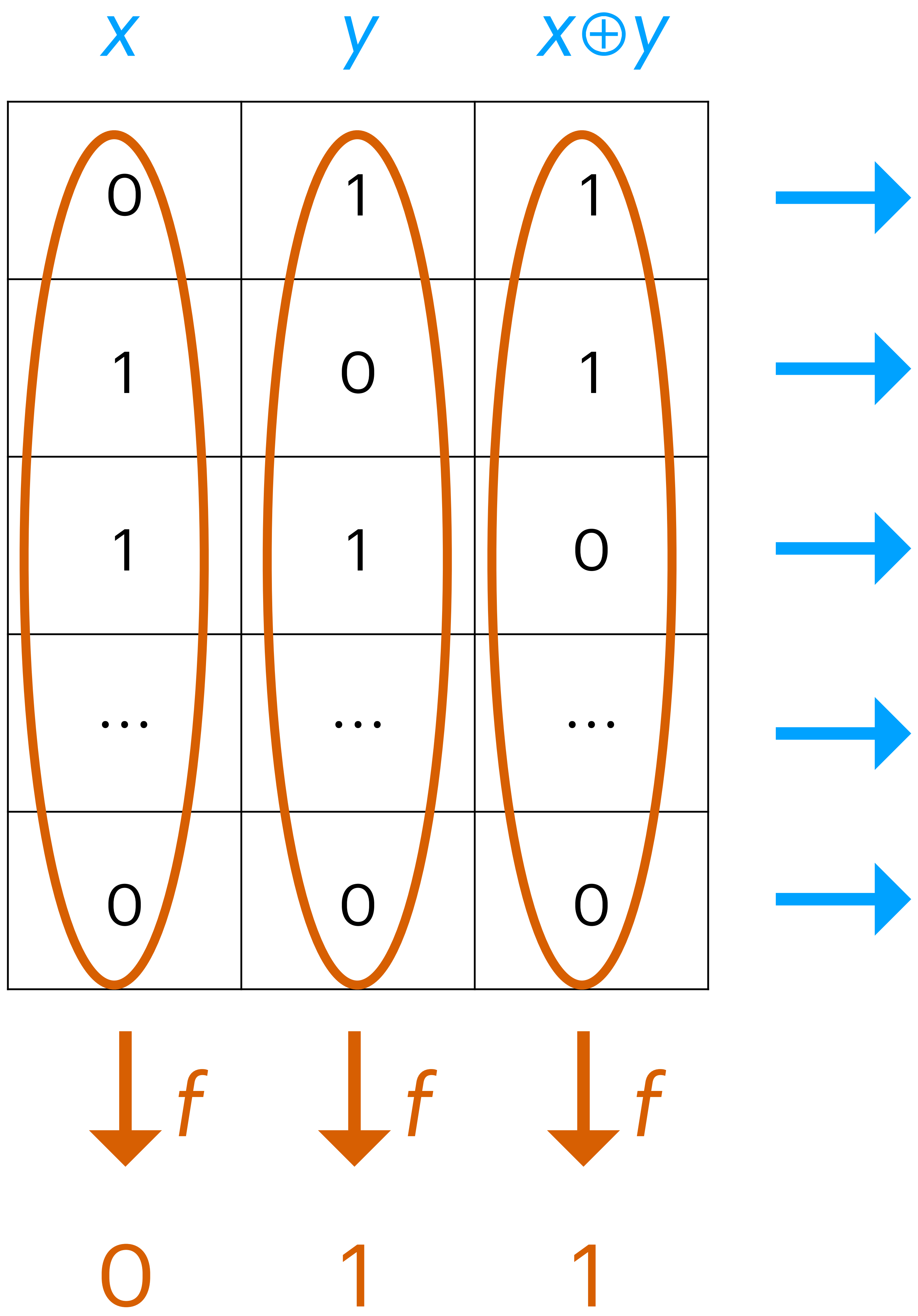

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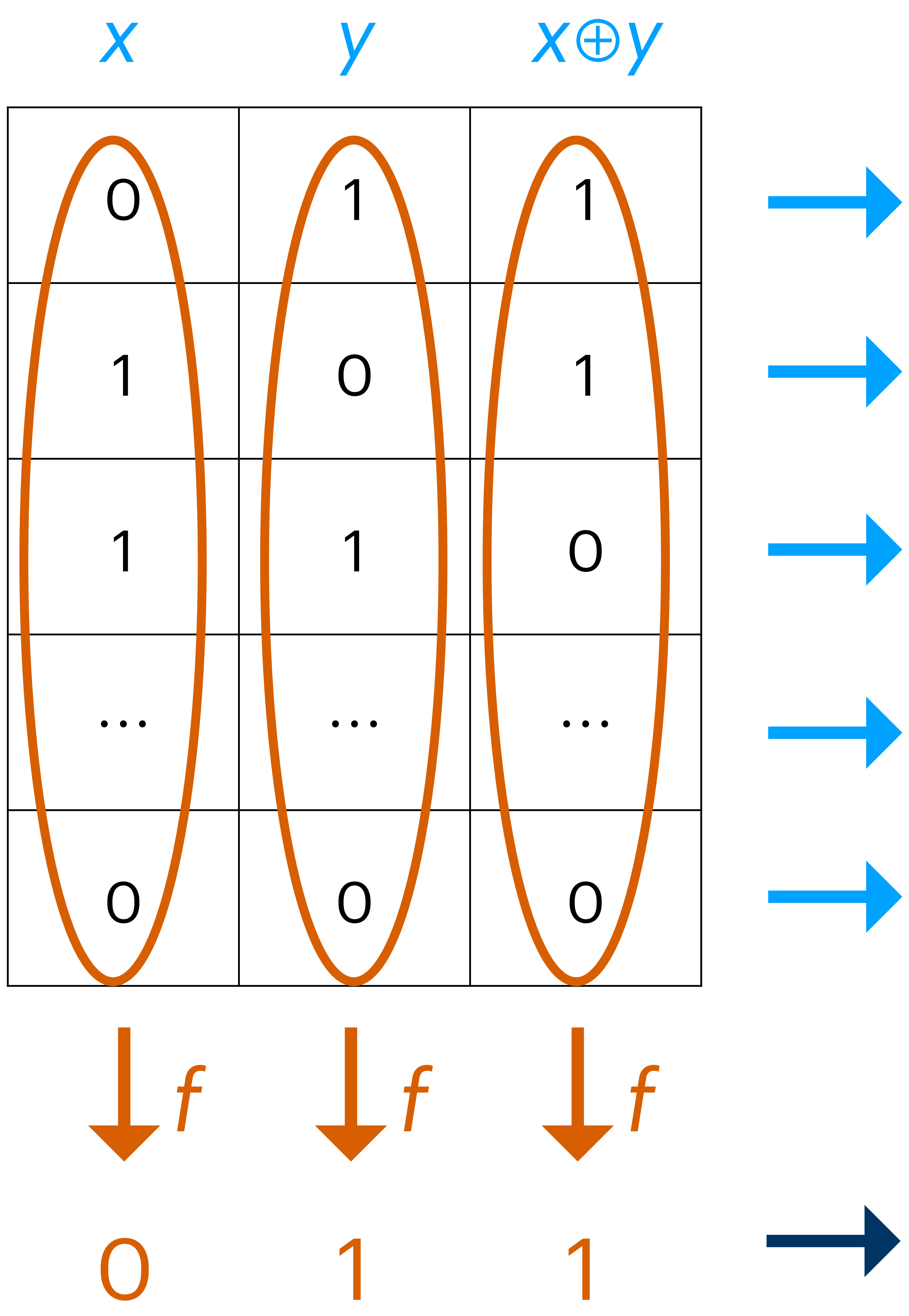

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Rows have even parity

Outcome must have even parity

Example: Even Parity

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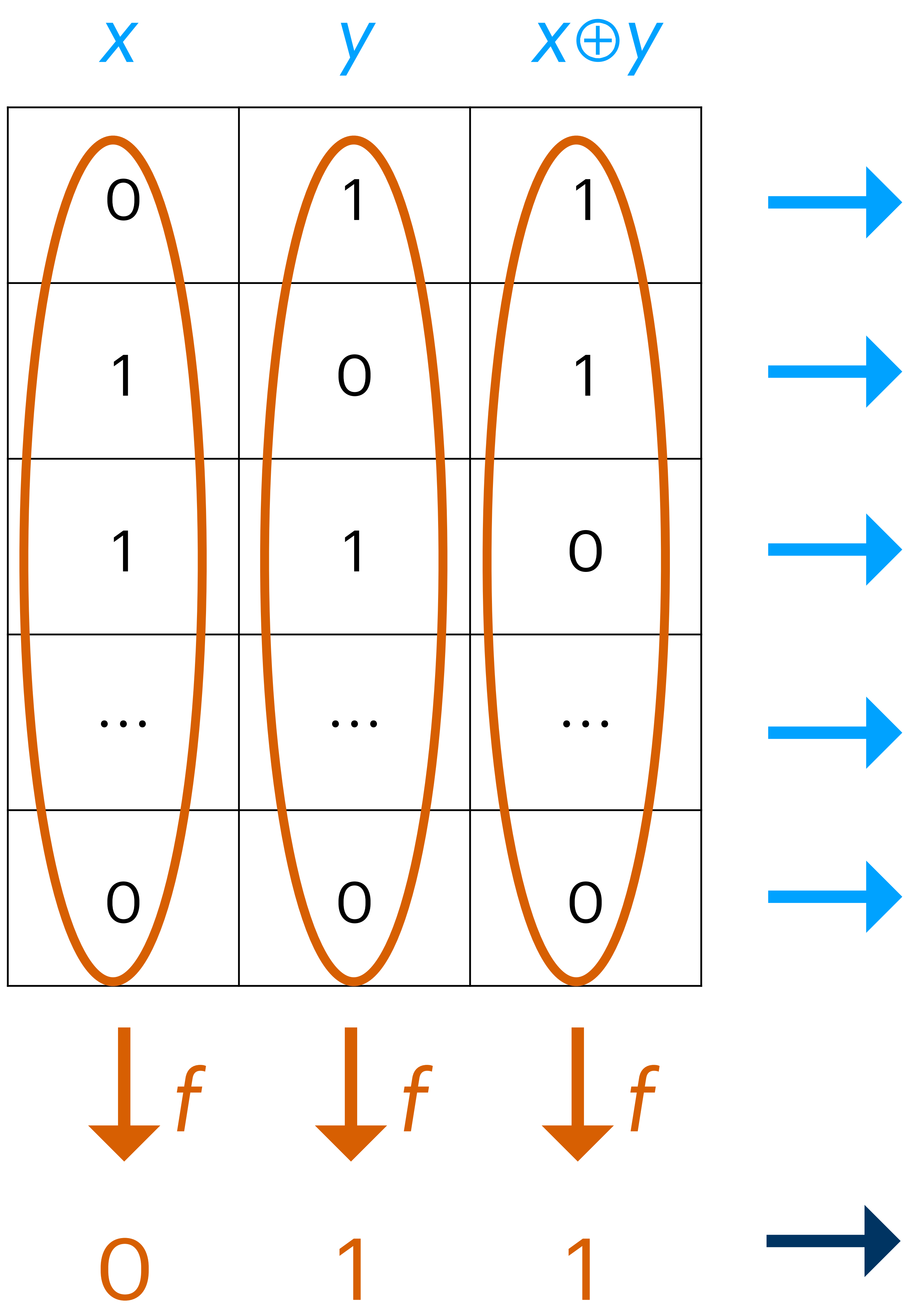
Polymorphisms:
XORs of rows



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Outcome must
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Example: AND

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Example: AND

A function $f\colon \{0,1\}^n \to \{0,1\}$ is *multiplicative* if $f(xy) = f(x)f(y)$

x	y	$x \wedge y$
0	1	0
1	0	0
1	1	1
…	…	…
0	0	0

Example: AND

A function $f\colon \{0,1\}^n \rightarrow \{0,1\}$ is *multiplicative* if $f(xy) = f(x)f(y)$

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1	0	0
1	1	1
...	...	...
0	0	0

Last coord
computes
AND

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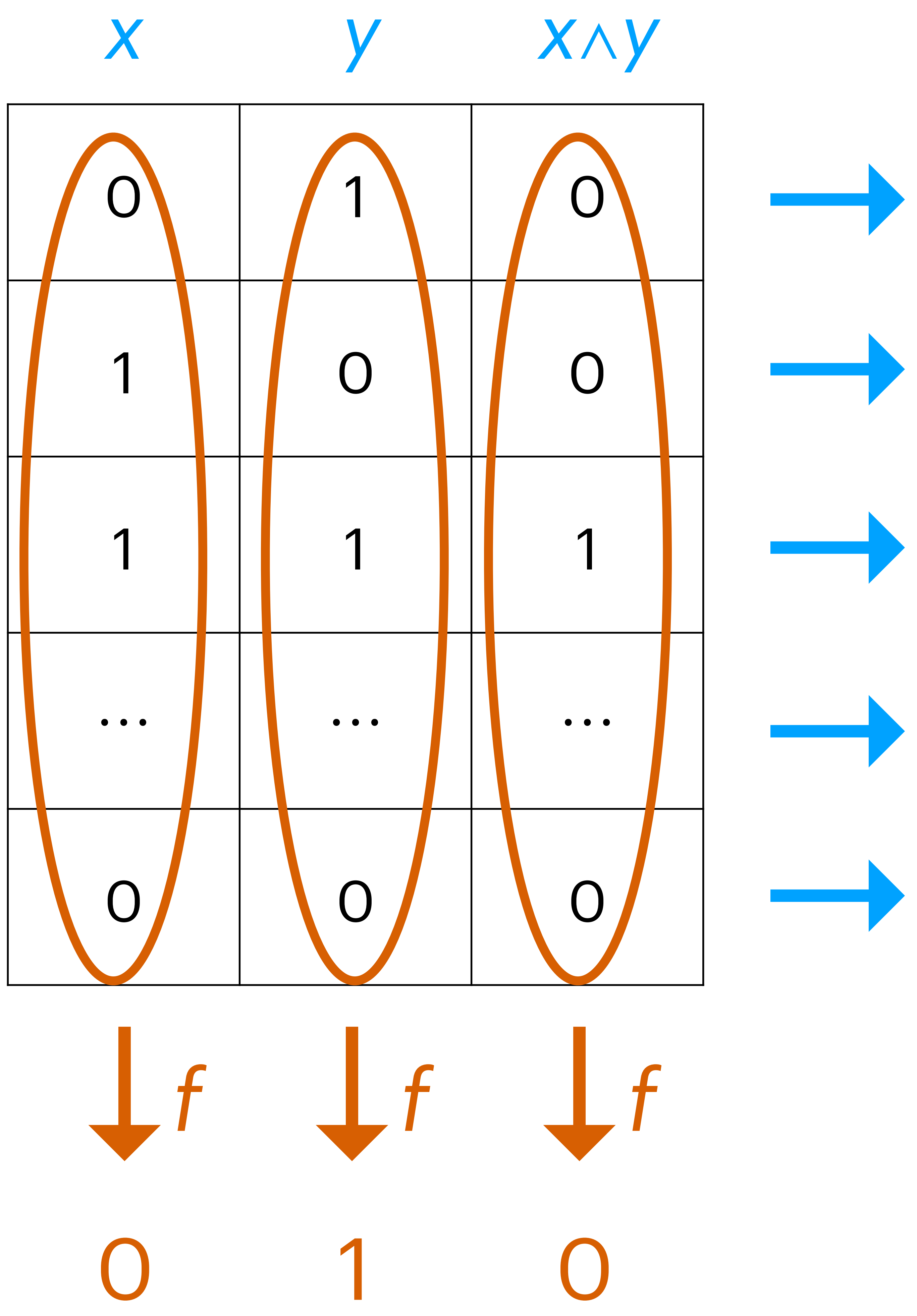

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1	0	0
1	1	1
…	…	…
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Last coord computes AND

Example: AND

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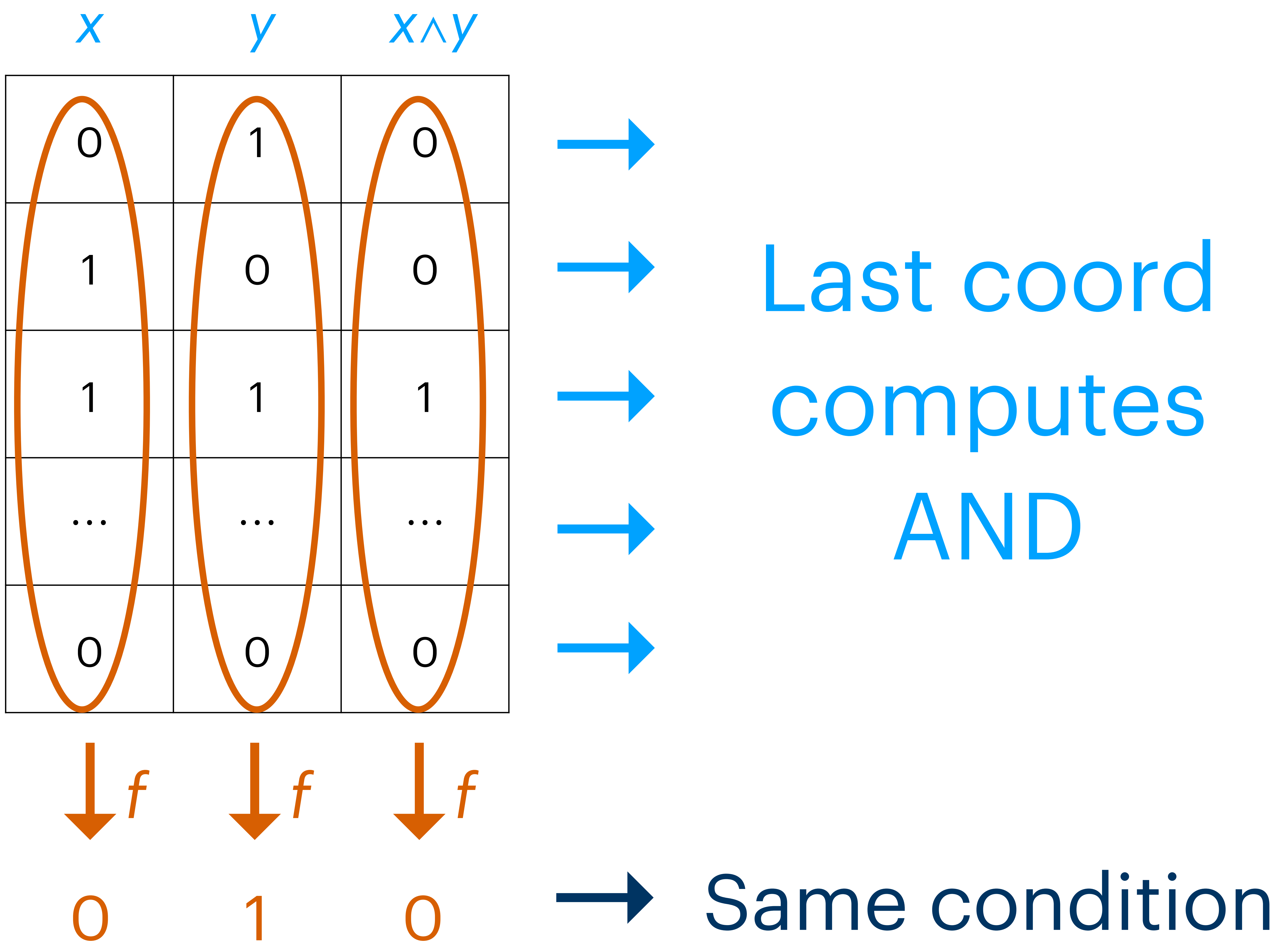

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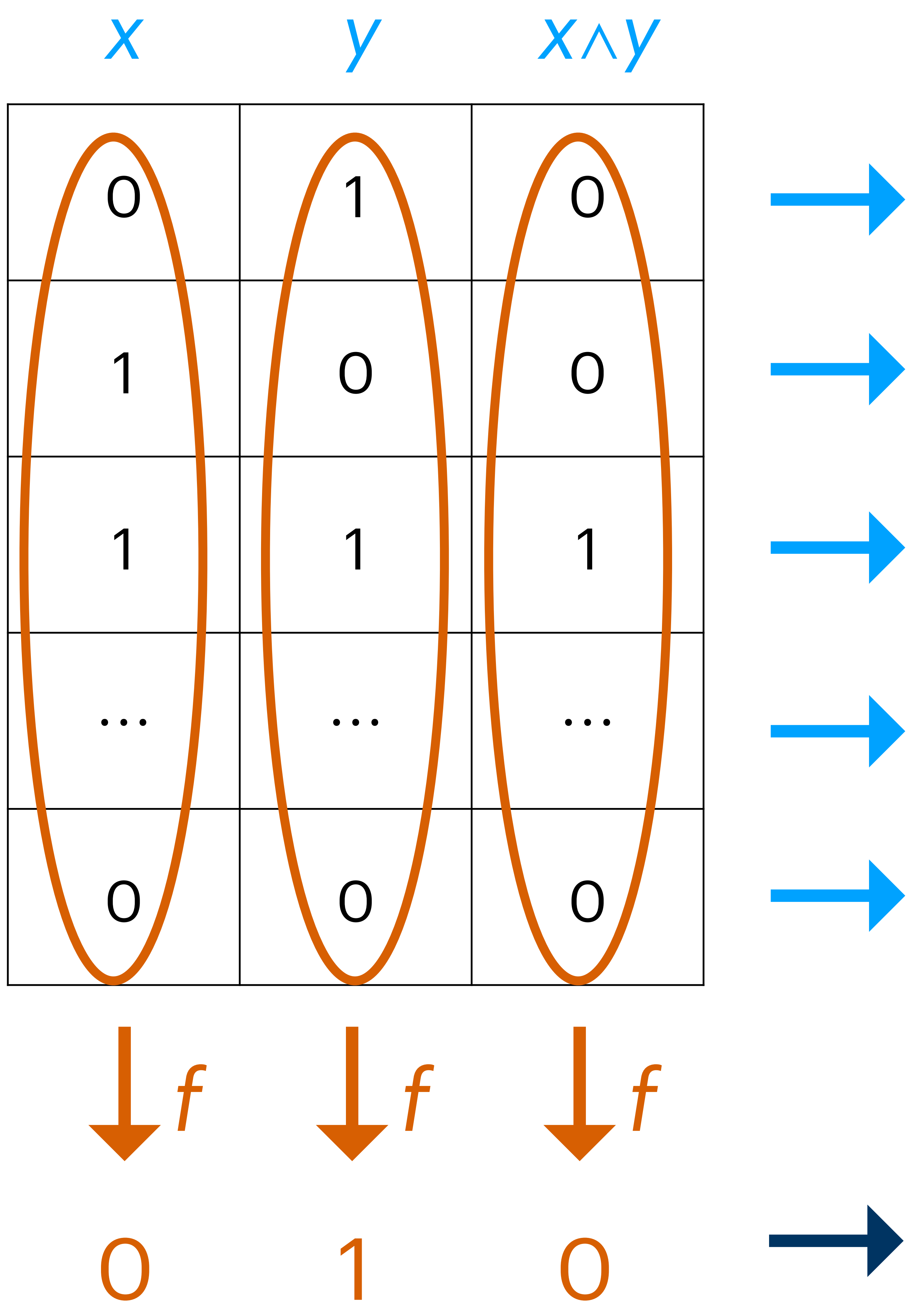
Last coord computes AND

Same condition

Example: AND

A function $f\colon \{0,1\}^n \to \{0,1\}$ is *multiplicative* if $f(xy) = f(x)f(y)$

Polymorphisms:
ANDs of rows, 0



x	y	$x \wedge y$
0	1	0
1	0	0
1	1	1
…	…	…
0	0	0
0	1	0

Last coord
computes
AND

→ Same condition

Example: NAND

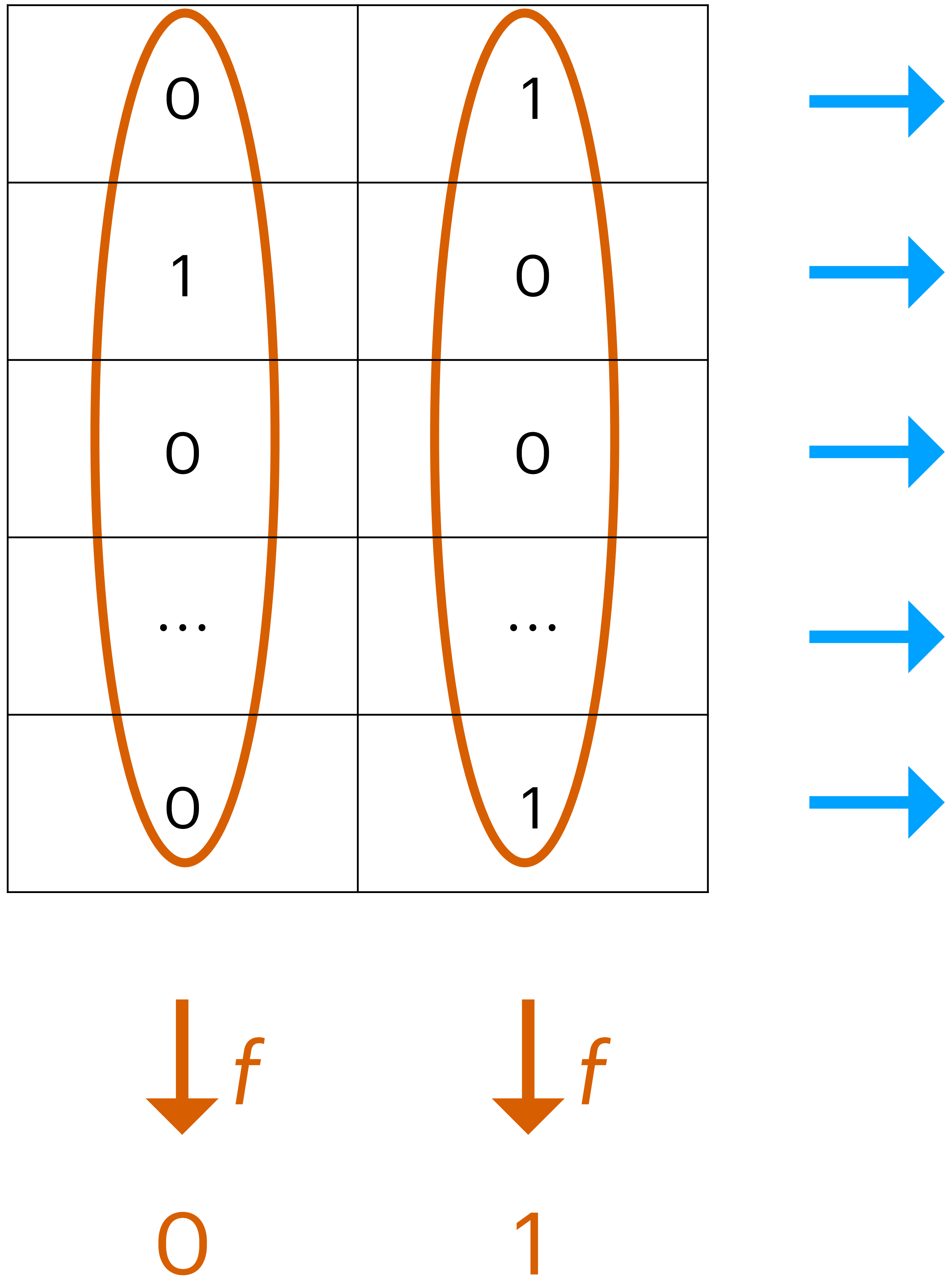
0	1
1	0
0	0
…	…
0	1

Example: NAND

0	1
1	0
0	0
…	…
0	1

Forbidden row:
11

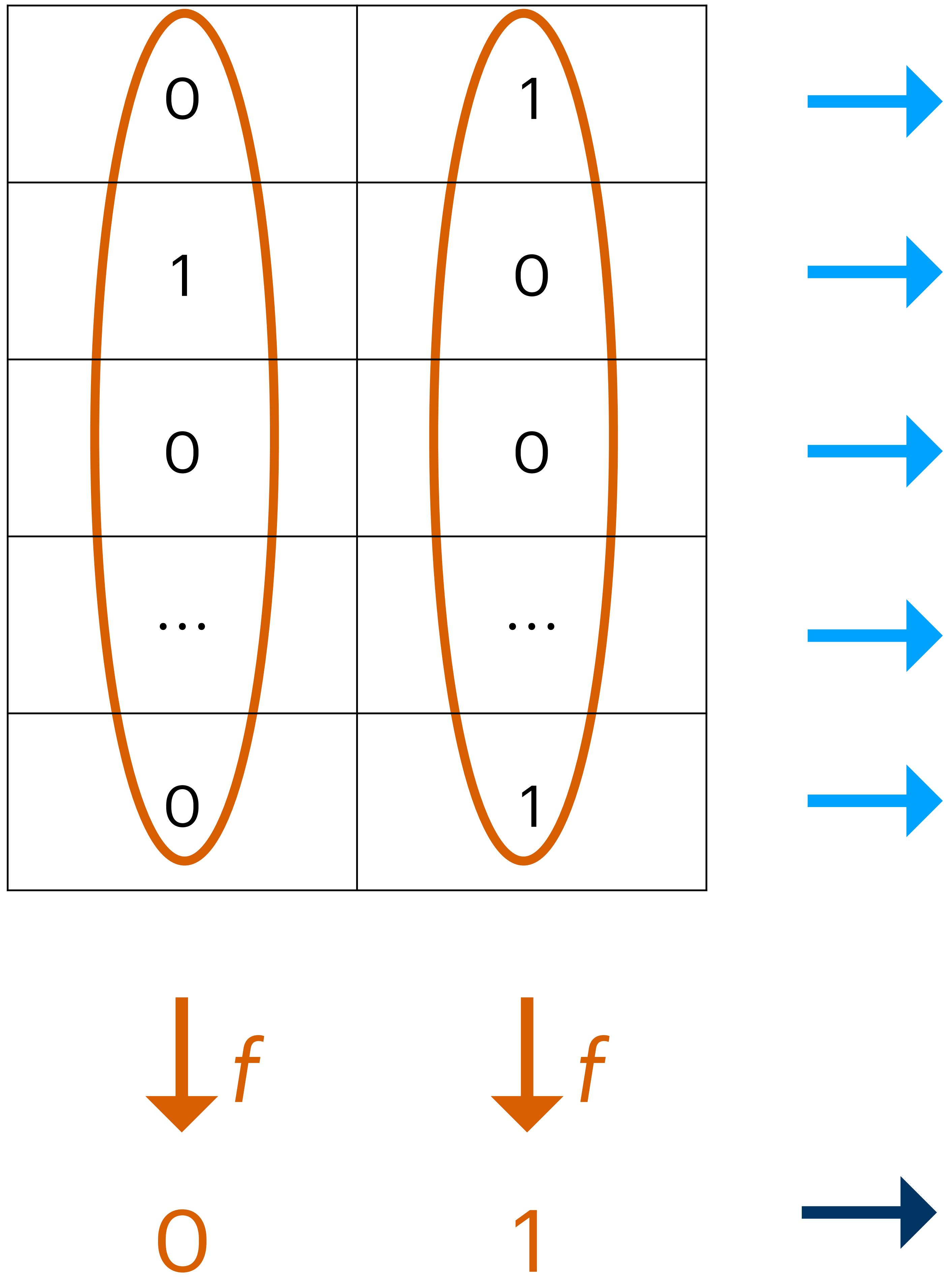
Example: NAND



0	1
1	0
0	0
…	…
0	1

Forbidden row:
11

Example: NAND



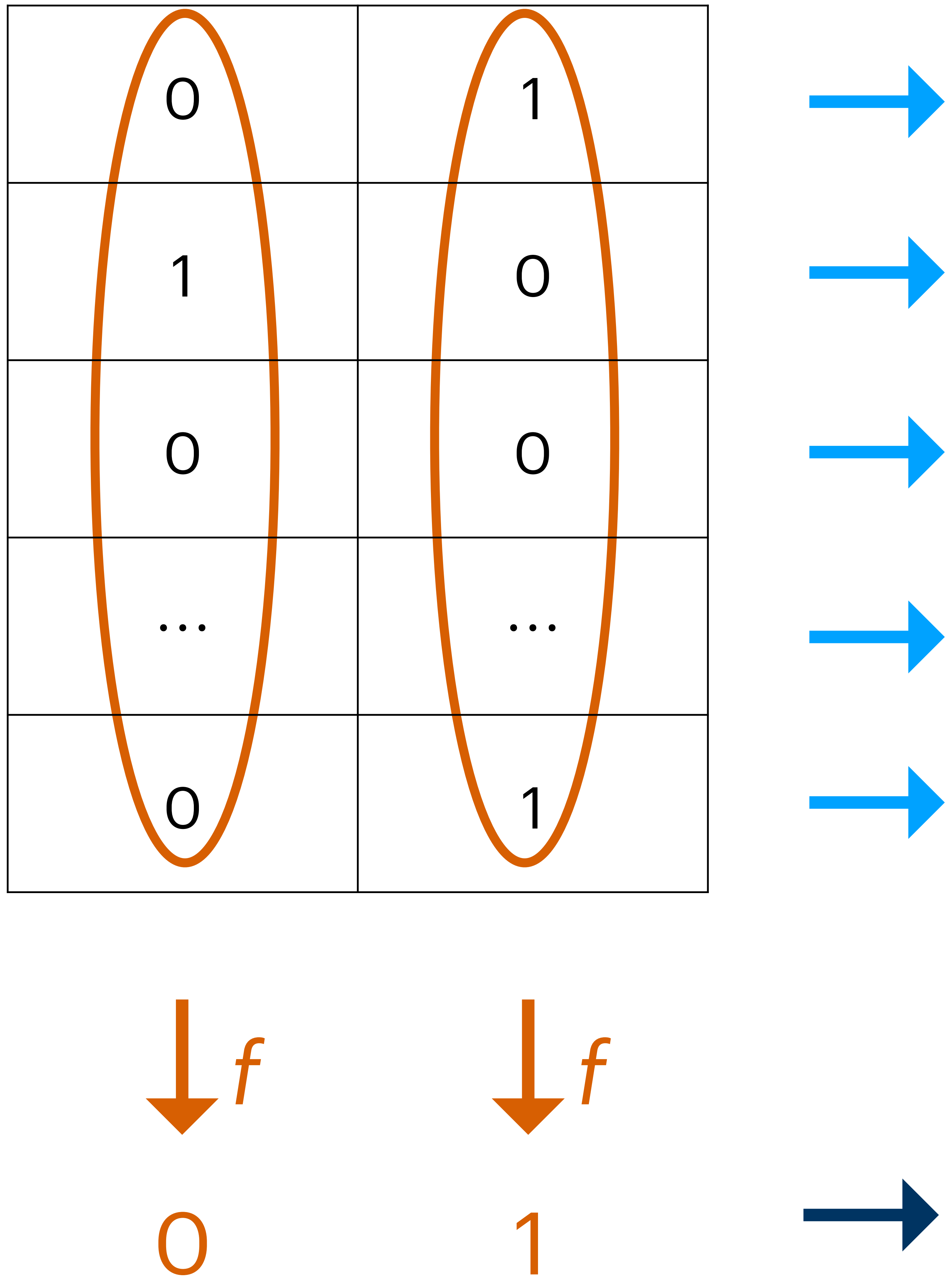
0	1
1	0
0	0
…	…
0	1

Forbidden row:
11

Outcome not 11

Example: NAND

Polymorphisms:
Intersecting families

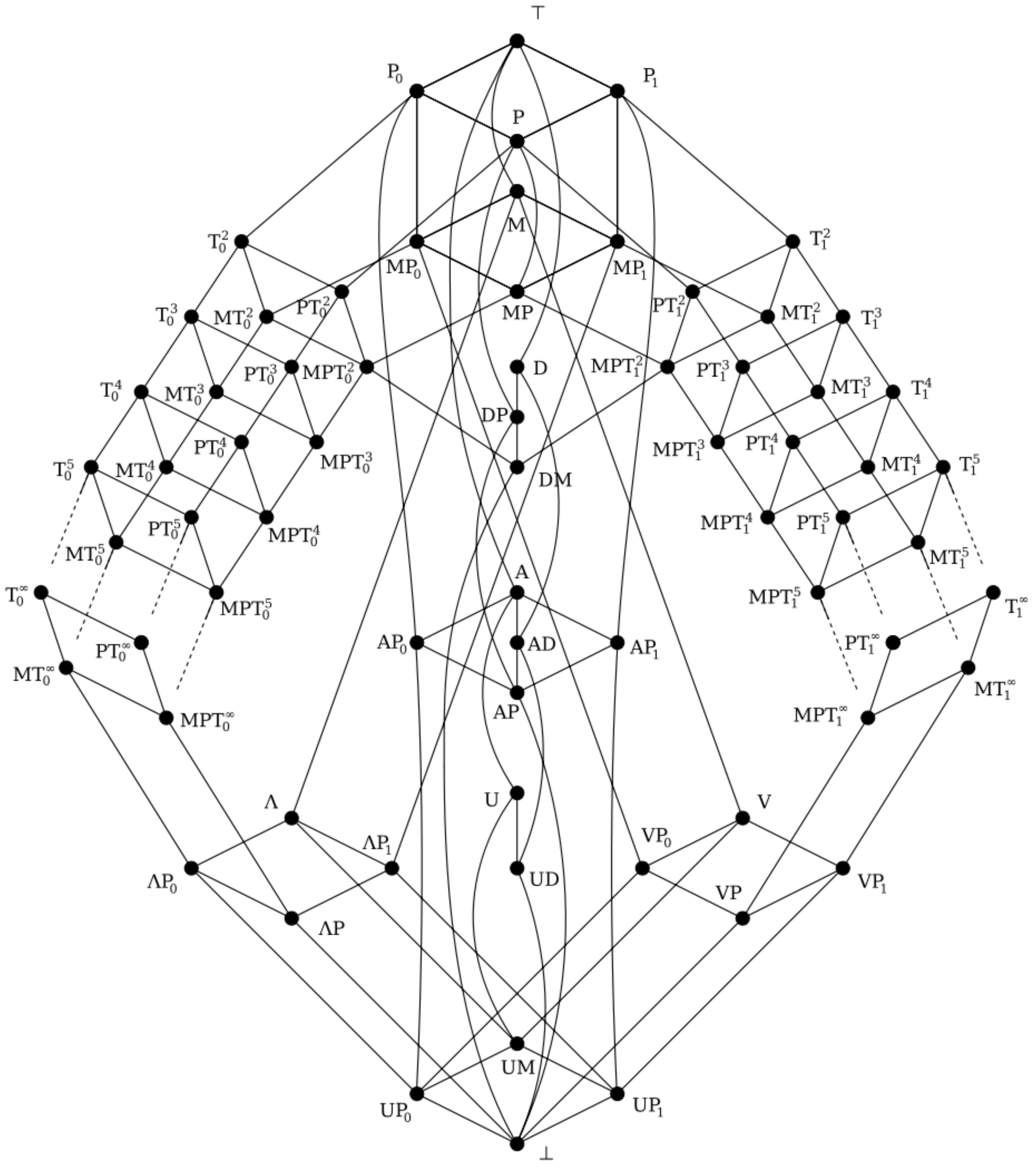


0	1
1	0
0	0
…	…
0	1

Forbidden row:
11

→ Outcome not 11

Post's Lattice



Truth-Functional Setting

Truth-Functional Setting

XOR function

0	1	$0 \oplus 1$
1	0	$1 \oplus 0$
1	1	$1 \oplus 1$
…	…	…
0	0	$0 \oplus 0$

Truth-Functional Setting

XOR function

0	1	$0 \oplus 1$
1	0	$1 \oplus 0$
1	1	$1 \oplus 1$
…	…	…
0	0	$0 \oplus 0$

AND function

0	1	$0 \wedge 1$
1	0	$1 \wedge 0$
1	1	$1 \wedge 1$
…	…	…
0	0	$0 \wedge 0$

Truth-Functional Setting

XOR function

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1	0	$1 \oplus 0$
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…	…	…
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AND function

0	1	$0 \wedge 1$
1	0	$1 \wedge 0$
1	1	$1 \wedge 1$
…	…	…
0	0	$0 \wedge 0$

Majority function

0	1	1	Maj(0,1,1)
1	1	1	Maj(1,1,1)
1	0	0	Maj(1,0,0)
…	…	…	…
0	0	0	Maj(0,0,0)

Truth-Functional Setting

XOR function

0	1	$0 \oplus 1$
1	0	$1 \oplus 0$
1	1	$1 \oplus 1$
…	…	…
0	0	$0 \oplus 0$

AND function

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Always have dictators, sometimes “antidictators”, sometimes constants

Truth-Functional Setting

XOR function

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…	…	…	…
0	0	0	Maj(0,0,0)

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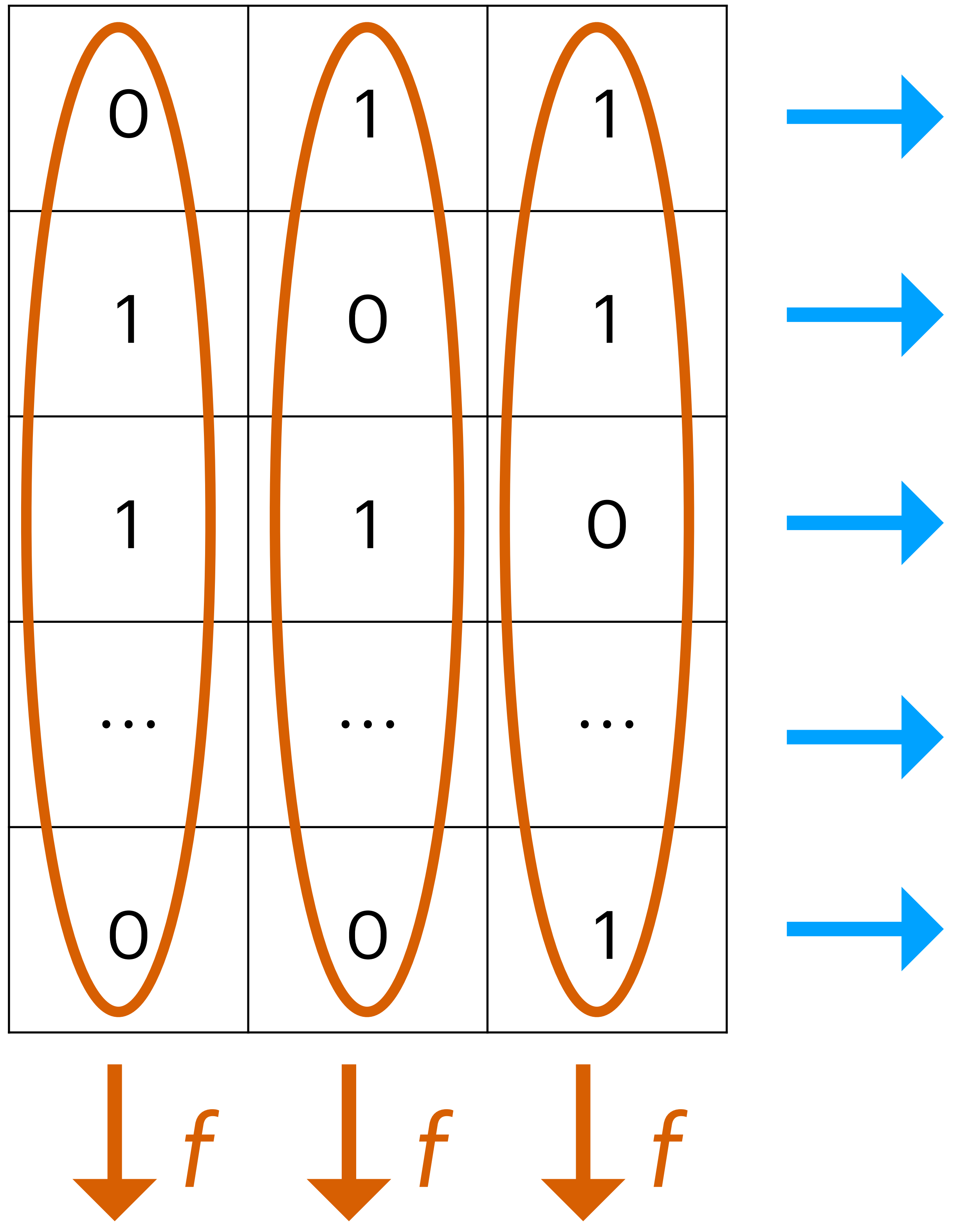
Dokow & Holzman: Other polymorphisms exist only for AND, XOR

Approximate Polymorphisms

Approximate Polymorphisms



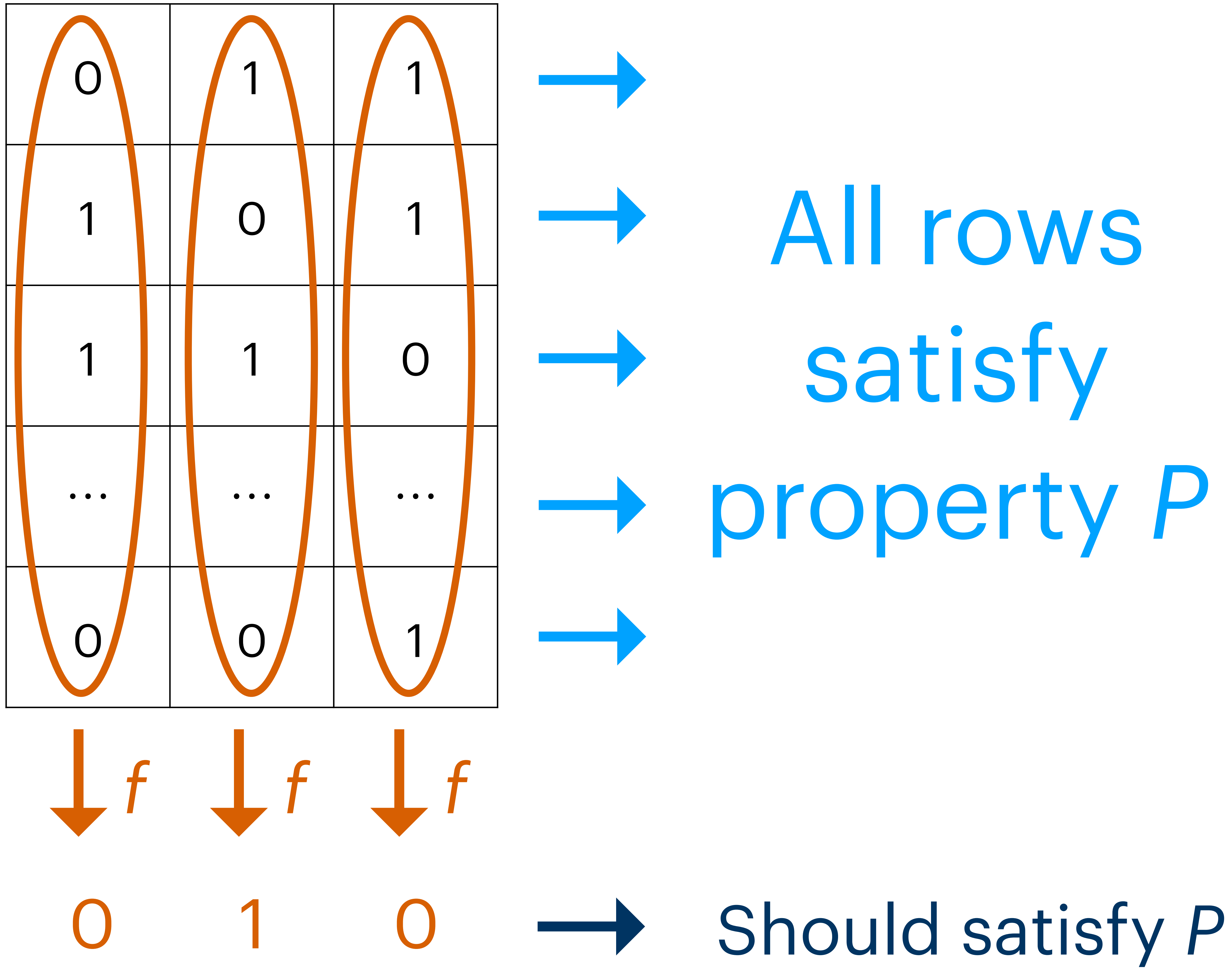
0	1	1
1	0	1
1	1	0
…	…	…
0	0	1

All rows satisfy property P

0 1 0 → Should satisfy P

Exact polymorphism

Approximate Polymorphisms



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1	0	1
1	1	0
…	…	…
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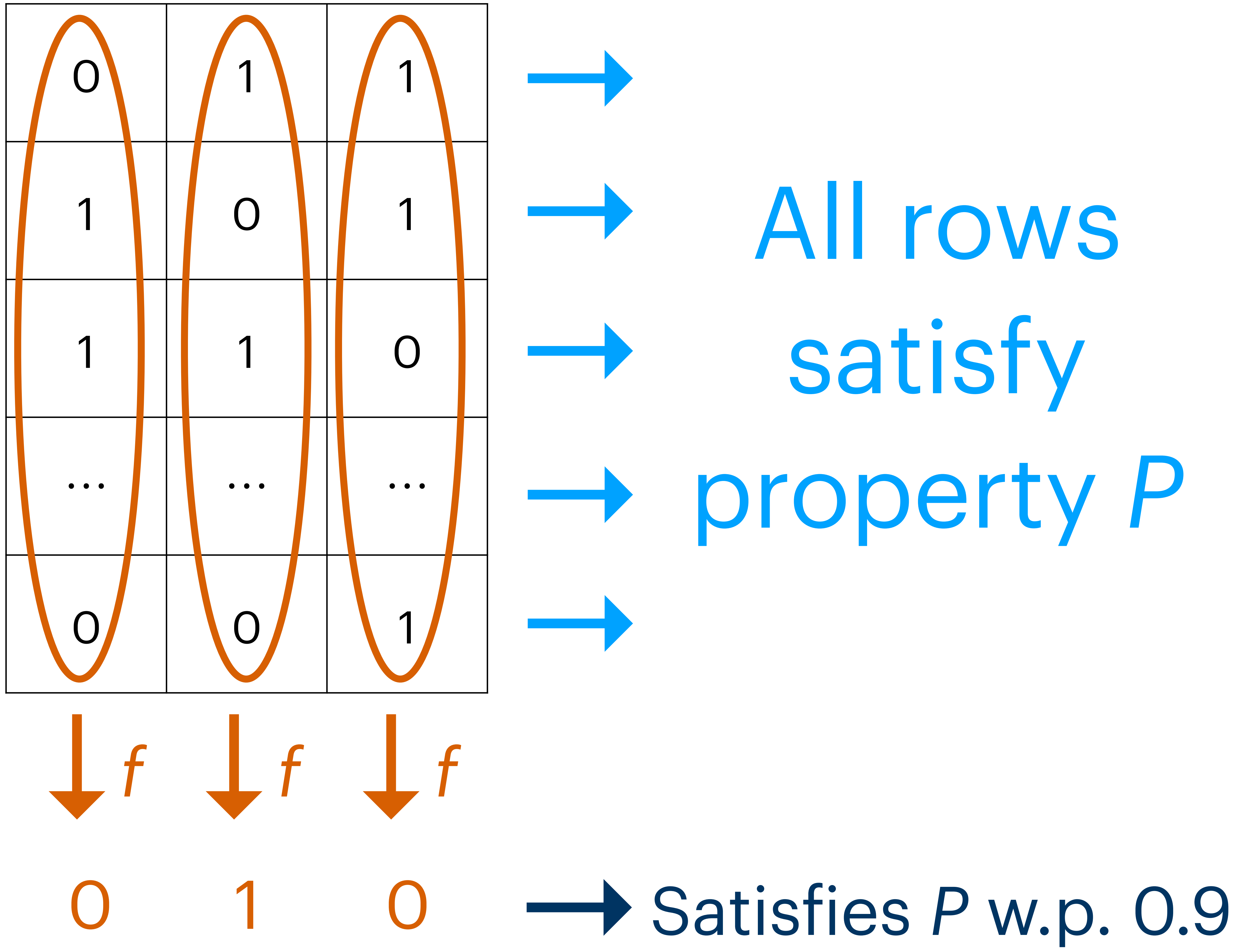
All rows satisfy property P

$\downarrow f$ $\downarrow f$ $\downarrow f$

0 1 0 → Should satisfy P

Exact polymorphism



0	1	1
1	0	1
1	1	0
…	…	…
0	0	1

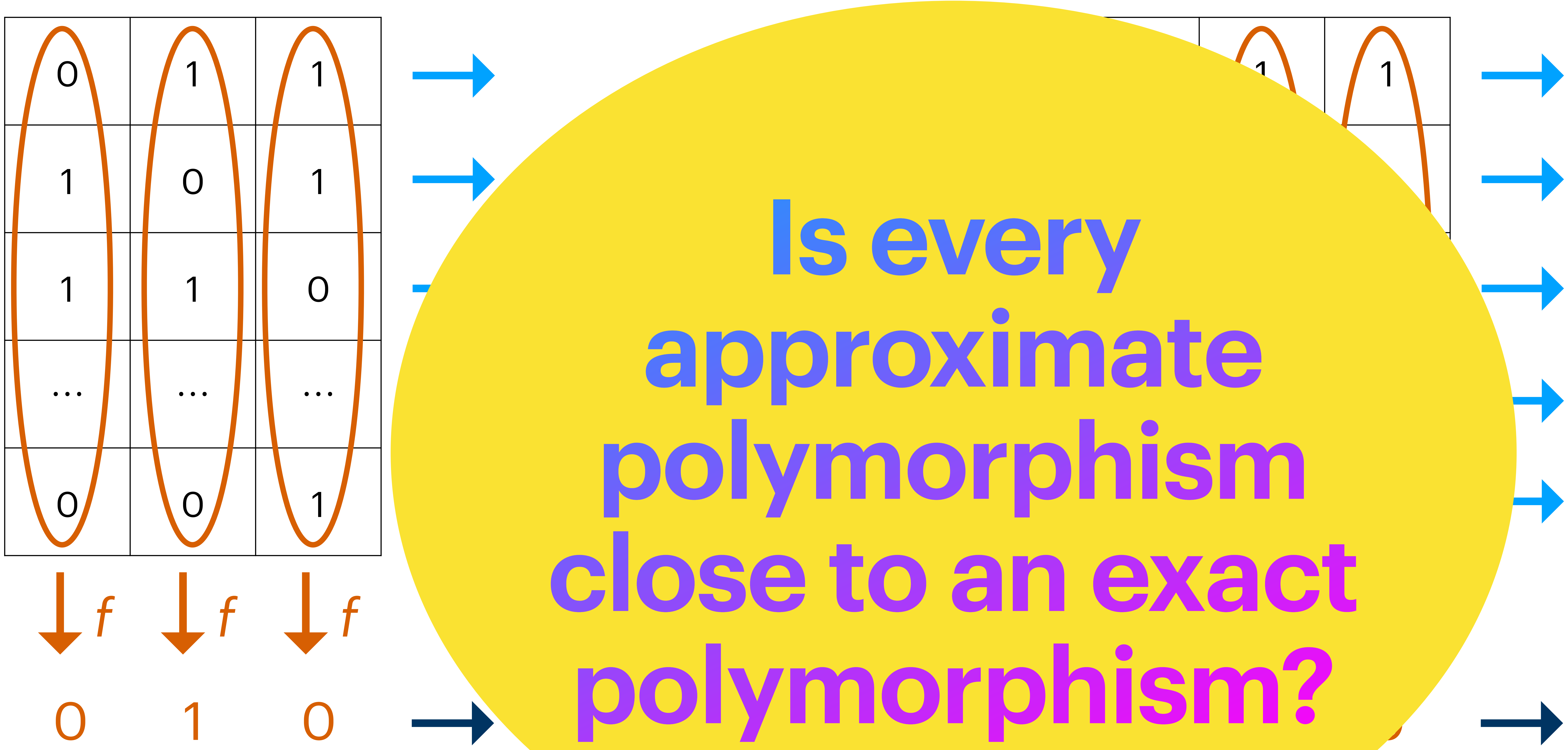
All rows satisfy property P

$\downarrow f$ $\downarrow f$ $\downarrow f$

0 1 0 → Satisfies P w.p. 0.9

Approx polymorphism

Approximate Polymorphisms



Is every approximate polymorphism close to an exact polymorphism?

All rows satisfy property P

Satisfies P w.p. 0.9

Exact polymorphi

pprox polymorphism

Examples of approximate polymorphisms

Not-All-Equal

0	1	1
1	0	1
1	1	0
…	…	…
0	0	1

Even Parity

0	1	1
1	0	1
1	1	0
…	…	…
0	0	0

NAND

0	1
1	0
0	0
…	…
0	1

AND function

0	1	0
1	0	0
1	1	1
…	…	…
0	0	0

Examples of approximate polymorphisms

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1	0	1
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0	1	1
1	0	1
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NAND

0	1
1	0
0	0
…	…
0	1

AND function

0	1	0
1	0	0
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…	…	…
0	0	0

Approx polymorphisms:
Dictators (*i*-th row)

(Kalai's theorem)

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1	0	1
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…	…	…
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Even Parity

0	1	1
1	0	1
1	1	0
…	…	…
0	0	0

Approx polymorphisms:
XORs of rows

(Linearity testing)

NAND

0	1
1	0
0	0
…	…
0	1

AND function

0	1	0
1	0	0
1	1	1
…	…	…
0	0	0

Examples of approximate polymorphisms

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0	1	1
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0	1	1
1	0	1
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…	…	…
0	0	0

Approx polymorphisms:
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NAND

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1	0
0	0
…	…
0	1

Approx polymorphisms:
Intersecting families

(Friedgut–Regev)

AND function

0	1	0
1	0	0
1	1	1
…	…	…
0	0	0

Examples of approximate polymorphisms

Not-All-Equal

0	1	1
1	0	1
1	1	0
…	…	…
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Approx polymorphisms:
Dictators (*i*-th row)

(Kalai’s theorem)

Even Parity

0	1	1
1	0	1
1	1	0
…	…	…
0	0	0

Approx polymorphisms:
XORs of rows

(Linearity testing)

NAND

0	1
1	0
0	0
…	…
0	1

Approx polymorphisms:
Intersecting families

(Friedgut–Regev)

AND function

0	1	0
1	0	0
1	1	1
…	…	…
0	0	0

Approx polymorphisms:
ANDs of rows, constant 0

(This work)

Examples of approximate polymorphisms

Not-All-Equal

0	1	1
1	0	1
1	1	0
…	…	…
0	0	1

Approx polymorphisms:
Dictators (*i*-th row)

(Kalai’s theorem)

Even Parity

0	1	1
1	0	1
1	1	0
…	…	…
0	0	0

Approx polymorphisms:
XORs of rows

(Linearity testing)

NAND

0	1
1	0
0	0
…	…
0	1

Approx polymorphisms:
Intersecting families

(Friedgut–Regev)

AND function

0	1	0
1	0	0
1	1	1
…	…	[illegible]
0	0	0

Improves on
Nehama 2010

Approx polymorphisms:
ANDs of rows, constant 0

(This work)

Background: Boolean Function Analysis

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Every function $f\colon \{\pm 1\}^n \to \{\pm 1\}$ has unique representation as multilinear poly

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Degree of f: degree of unique representation (as polynomial)

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Every function f: $\{\pm 1\}^n \to \{\pm 1\}$ has unique representation as multilinear poly

Degree of f: degree of unique representation (as polynomial)

Noise operator T_ρ multiplies degree d monomials (“level d“) by ρ^d

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Every function $f\colon \{\pm 1\}^n \to \{\pm 1\}$ has unique representation as multilinear poly

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Noise operator T_ρ multiplies degree d monomials (“level d“) by ρ^d

Constant coefficient is expectation of f

Background: Boolean Function Analysis

Every function $f\colon \{\pm 1\}^n \to \{\pm 1\}$ has unique representation as multilinear poly

Degree of f: degree of unique representation (as polynomial)

Noise operator T_ρ multiplies degree d monomials (“level d“) by ρ^d

Constant coefficient is expectation of f

Important observation: different monomials are *orthogonal*

Simpler example

Majority function

0	1	1	1
1	1	1	1
1	0	0	0
…	…	…	…
0	0	0	0

→

Last coord computes Majority

Polymorphisms:
Dictators (*i*-th row)
Constant functions

Polymorphisms of Majority

A function $f\colon \{\pm 1\}^n \to \{\pm 1\}$ is a polymorphism of Majority if

$$f(\mathsf{Maj}(x_1, y_1, z_1), \ldots, \mathsf{Maj}(x_n, y_n, z_n)) = \mathsf{Maj}(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n), f(z_1, \ldots, z_n))$$

Polymorphisms of Majority

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Fix x, average over y, z: $\quad T_{1/2}f(x) = \frac{1-\mu^2}{2}f(x) + \mu, \qquad \mu = \mathbb{E}[f]$

Polymorphisms of Majority

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Everything also holds approximately, using FKN theorem!

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Cannot directly compare biased and unbiased Fourier expansions!

The two expansions depend on different parts of f.

However, can read Fourier expansion of $T_\downarrow f$

from *biased* Fourier expansion of f!

Approximate polymorphisms of AND

Starting point: $T_{\downarrow} f \approx \mu f$, where $\mu = \mathbb{E}[f]$.

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Suggests solving generalized eigenvalue problem

$$T_\downarrow g = \lambda h$$

where $g \colon \{0,1\}^n \to [0,1]$ and $h \colon \{0,1\}^n \to \{0,1\}$.

Generalized eigenvalue problem

Solve $T_{\downarrow}g(x) = \lambda h(x)$ for $g\colon \{0,1\}^n \to [0,1]$ and $h\colon \{0,1\}^n \to \{0,1\}$.

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If $x_1 = \cdots = x_\ell = 1$ then $y_1 = \cdots = y_\ell = 1$ w.p. $2^{-\ell}$.

Otherwise, $y_1 \wedge \cdots \wedge y_\ell = 0$ *always*.

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If $x_1 = 1$ then y_1 is uniformly random, so $y_1 \oplus y_2$ is uniformly random.

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General case: g is $c \cdot$ AND of XORs, h is AND of ORs, $\lambda = c \cdot 2^{-\text{\#factors}}$.

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Can rule out unexpected solutions since $\lambda \approx \mathbb{E}[h]$.

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First step: h is monotone.

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LP duality: argument automatically extends to $T_{\downarrow}g \approx \lambda h$!

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- Same for Majority on any odd number of inputs.
- Ongoing work: many more functions!

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Proof is somewhat different!

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- If $\Pr[f(x \oplus y) = f(x) \oplus f(y)] \geq \frac{1}{2} + \varepsilon$ then f *correlates* with exact polymorphism.

  Does a similar statement hold for AND?

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Schaefer's theorem:
If all predicates have one of the following polymorphisms, in P:
constant 0, constant 1, AND, OR, Majority, XOR
Otherwise, NP-complete.

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Recently extended to non-binary domains (Dichotomy Theorem).