Spectral methods in extremal combinatorics

Yuval Filmus

November 4, 2013

1 Introduction

The area of *analysis of boolean functions* has become commonplace in theoretical computer science. In this short talk, we would like to explain one application outside of computer science, namely to *extremal combinatorics* of the *Erdős–Ko–Rado* variety.

2 Erdős–Ko–Rado theory

The celebrated Erdős–Ko–Rado theorem [9], proved at 1938 but published only at 1961, states the following. Suppose $\mathcal{F} \subseteq \binom{[n]}{k}$ is an *intersecting family* of sets. This means that (1) $\mathcal{F}$ consists of subsets of size k of the ground set $[n] = \{1, \ldots, n\}$, and (2) any two sets in $\mathcal{F}$ contain at least one point in common. Then:

Upper bound: If $k \leq n/2$ then $|\mathcal{F}| \leq \binom{n-1}{k-1}$.

Uniqueness: If $k < n/2$ and $|\mathcal{F}| = \binom{n-1}{k-1}$ then $\mathcal{F}$ is a 1-star.

Stability: If $k < n/2$ and $|\mathcal{F}| \geq (1-\epsilon)\binom{n-1}{k-1}$ then there is an element contained in $1 - O(\epsilon)$ of the sets in $\mathcal{F}$.

Here a 1-*star* is a family of the form $\{S \in \binom{[n]}{k} : x \in S\}$ for some $x \in [n]$. The last part (stability) is not found in the original paper, and is essentially proved in Frankl [11].

The Erdős–Ko–Rado paper opened up an entire research are in extremal combinatorics. Their original result was extended in various ways:

Strong stability: Hilton and Milner [16, 12] showed that if $k \leq n/2$ and $|\mathcal{F}| > \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$ then $\mathcal{F}$ is contained in a star.

Variants: The notion of being intersecting has been extended:

- Ahlswede and Khachatrian [1, 2] found the maximum t-*intersecting families* (families in which any two sets contain at least t points in common) which are subsets of $\binom{[n]}{k}$.

- Pyber [21] proved the *cross-intersecting* version of the Erdős–Ko–Rado theorem: for $k \leq n/2$, if $\mathcal{F}, \mathcal{G} \subseteq \binom{[n]}{k}$ are *cross-intersecting* (every set in $\mathcal{F}$ intersects every set in $\mathcal{G}$) then $|\mathcal{F}||\mathcal{G}| \leq \binom{n-1}{k-1}^2$.
- Frankl [10] showed that if $k \leq (r-1)/r \cdot n$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ is *r-wise intersecting* (any r sets in $\mathcal{F}$ intersect) then $|\mathcal{F}| \leq \binom{n-1}{k-1}$.

Different domains: The theorem has been generalized to different domains, such as:

- Deza and Frankl [4] showed that every intersecting family of permutations on n points contains at most $(n-1)!$ permutations; two permutations intersect if they agree on the image of at least one point. This was extended to t-intersecting families by Ellis, Friedgut and Pilpel [8].
- Frankl and Wilson [13] showed that if $\mathcal{F}$ is a family of k-dimensional subspaces of $\mathrm{GF}(q)^n$ whose pairwise intersections have dimension at least t then for $n \geq 2k$, $|\mathcal{F}| \leq \begin{bmatrix} n-t \\ k-t \end{bmatrix}_q$, which is the number of $(k-t)$-dimensional subspaces of $\mathrm{GF}(q)^{n-t}$.
- Ellis, Filmus and Friedgut [7] showed that every *odd-cycle-intersecting* family of subgraphs of K_n contains at most $2^{\binom{n}{2}-3}$ graphs. This is a family in which any two graphs contain an odd cycle in common.

3 Katona's circle argument

There are many ways to prove the Erdős–Ko–Rado theorem. The simplest one is due to Katona [18]. Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be an intersecting family, where $k \leq n/2$. Consider the n sets

$$\{1, \ldots, k\}, \{2, \ldots, k+1\}, \ldots, \{k+1, 1, \ldots, k-1\}.$$

Suppose that $\{k, \ldots, 2k-1\} \in \mathcal{F}$. Since $\mathcal{F}$ is intersecting, the only other sets that could be in $\mathcal{F}$ are $\{a, \ldots, a+k-1\}$ for $1 \leq a \leq 2k-1 < n$. Let $a_{\min}, a_{\max}$ be the minimal and maximal such a. Since $\mathcal{F}$ is intersecting, $a_{\max} - a_{\min} \leq k-1$, and so $\mathcal{F}$ contains k sets out of the listed n sets.

Denote the collection of sets above by $C_{1 \cdots n}$. In general, we can consider collections C_π for arbitrary $\pi \in S_n$, and these collections partition $\binom{[n]}{k}$: indeed, the relation of *being related by cyclic rotation* is an equivalence relation, and we can divide $\binom{[n]}{k}$ into $\binom{n}{k}/n$ equivalence classes. The family $\mathcal{F}$ can contain at most k sets out of each of these equivalence classes, hence $|\mathcal{F}| \leq (k/n)\binom{n}{k} = \binom{n-1}{k-1}$. This gives the upper bound. Uniqueness can be derived using some more effort.

Katona's proof idea works in some other circumstances, for example:

- Frankl [10] showed that if $k \leq (r-1)/r \cdot n$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ is r-wise intersecting then $|\mathcal{F}| \leq \binom{n-1}{k-1}$, using essentially the same argument.
- Deza and Frankl [4] showed that an intersecting family of permutations in S_n contains at most $(n-1)!$ permutations, by considering cyclic rotations of permutations.

However, the proof does not extend to other situations, such as the Ahlswede–Khachatrian theorem or triangle-intersecting families.

4 Lovász–Hoffman method

4.1 Erdős–Ko–Rado proof

Lovász [20], in his paper describing his *theta* function, proved the Erdős–Ko–Rado theorem using a spectral method, which can be traced back to Hoffman [17]. The idea is to consider the Kneser graph $\mathrm{Kn}(n,k)$. In this graph the set of vertices are $\binom{[n]}{k}$, and two vertices are connected in the corresponding sets don't intersect. Since $k \leq n/2$, the graph is not empty. The corresponding adjacency matrix A is symmetric and so has a basis of orthonormal eigenvectors. Since A is regular, the maximal eigenvector is $\mathbf{1}$, which has an eigenvalue of $\lambda = \binom{n-k}{k}$. It turns out that the minimal eigenvalue is $\lambda_{\min} = -\binom{n-k-1}{k-1}$.

Now consider an intersecting family $\mathcal{F} \subset$ and its corresponding characteristic function f. We can decompose f into its component along $\mathbf{1}$ and its component in $\mathbf{1}^\perp$. (Our inner product is $\langle g,h\rangle = \mathbb{E}_S g(S)h(S)$.) The component along $\mathbf{1}$ is $f_0 = \mathbb{E}[f]\mathbf{1}$, and we have $\|f\|^2 = \|f_0\|^2 + \|f-f_0\|^2$. Since $\|f\|^2 = \mathbb{E}[f^2] = \mathbb{E}[f]$,

$$\begin{aligned} 0 = \langle f, Af\rangle &\geq \lambda\|f_0\|^2 + \lambda_{\min}\|f-f_0\|^2 \\ &= \lambda\mathbb{E}[f]^2 + \lambda_{\min}(\mathbb{E}[f] - \mathbb{E}[f]^2). \end{aligned}$$

Simple algebra now gives *Hoffman's bound*:

$$\mathbb{E}[f] \leq \frac{-\lambda_{\min}}{\lambda - \lambda_{\min}}. \tag{1}$$

Substituting the values of λ and $\lambda_{\min}$, we easily obtain $\mathbb{E}[f] \leq k/n$.

Furthermore, when $\mathbb{E}[f] = k/n$, the argument shows that $f - f_0$ must belong to the eigenspace of $\lambda_{\min}$, and so f itself must belong to the span of the eigenspaces of λ and $\lambda_{\min}$. When $k < n/2$, this subspace is spanned by 1-stars, and a short argument implies that $\mathcal{F}$ is a 1-star.

4.2 Wilson's extension

Wilson [22] extended Lovász's argument to show that when $(k-t+1)/n \leq 1/(t+1)$, a t-intersecting family $\mathcal{F} \subseteq \binom{[n]}{k}$ contains at most $\binom{n-t}{k-t}$ sets. One natural attempt would be to consider the graph G_{t-1} in which the vertex set is $\binom{[n]}{k}$ and two vertices are connected if the corresponding sets have fewer than t elements in common. However, this graph has the "wrong" eigenvalues.

Instead, we look at the subspace of matrices spanned by the adjacency matrices of the graphs $G_0, \ldots, G_{t-1}$. Each matrix A in this subspace satisfies the crucial relation $f'Af = 0$ for every t-intersecting family $\mathcal{F}$ and its characteristic function f. By carefully choosing the matrix A, Wilson was able to arrange for (1) to give the correct bound $|\mathcal{F}| \leq \binom{n-t}{k-t}$.

Furthermore, when $(k - t + 1)/n < 1/(t + 1)$ and $|\mathcal{F}| = \binom{n-t}{k-t}$, the same reasoning as before showes that f belongs to the span of t-stars, and a short argument shows that it must be a t-star.

4.3 General formulation

We can formulate Hoffman's method more generally. Suppose we have a graph G, and want to obtain an upper bound on the maximum independent set in G. Choose any symmetric matrix A whose rows and columns are indexed by vertices of G, such that:

1. all rows in A sum to the same value λ,
2. $A_{ij} = 0$ whenever i, j are not connected in G.

If $\mathcal{F}$ is any independent set then its characteristic function f satisfies $\langle f, Af \rangle = 0$, and so Hoffman's bound (1) applies, with $\lambda_{\min}$ being the minimal eigenvalue of A.

The best bound obtained in this way can be termed the "Hoffman function" of the graph. The Lovász theta function is a similar bound which is always at least as tight as Hoffman's function. In many cases, Hoffman's bound is already tight. Indeed, this approach has been applied in several other situations:

- Frankl and Wilson [13] used Hoffman's bound to prove the Erdős–Ko–Rado theorem for vector spaces.
- Friedgut [14] used Hoffman's bound to prove a weighted analog of Wilson's result.
- Ellis, Friedgut and Pilpel [8] used Hoffman's bound to prove the Erdős–Ko–Rado theorem for t-intersecting families of permutations.
- Ellis, Filmus and Friedgut [7] used Hoffman's bound to show that odd-cycle-intersecting families of graphs on n points contain at most $2^{\binom{n}{2}-3}$ graphs.

In the sequel, we concentrate on the latter result.

5 Odd-cycle-intersecting families of graphs

In 1976, Simonovits and Sós asked the following question: *Suppose $\mathcal{F}$ is a family of subgraphs of K_n (the complete graph on n points) such that any two graphs in the family have some triangle in common. How big can $\mathcal{F}$ be?* They conjectured that the maximum family is a *triangle-junta*, consisting of all graphs containing a fixed triangle. This family contains 1/8 of all graphs. Chung, Graham, Frankl and Shearer [3] used Shearer's lemma in 1986 to give an upper bound of 1/4, and this was the best until the result of Ellis, Filmus and Friedgut [7].

The result of Chung, Graham, Frankl and Shearer also held for *odd-cycle-intersecting* families, in which any two graphs can contain any odd cycle in common; equivalently, the intersection of any two graphs is non-bipartite. Their result also holds for *odd-cycle-agreeing* families, in which the *intersection* of two sets $A \cap B$ (in this case, sets of edges) is replaces by their *agreement* $\overline{A \oplus B}$. As they showed, this is an example of a general phenomenon: in many cases, bounds on intersecting families transfer to agreeing families.

5.1 Feasible matrices

We now turn to the proof of Ellis, Filmus and Friedgut. We will be ambitious, trying to prove an upper bound on odd-cycle-agreeing families. Our goal is to find a symmtric matrix A, indexed by subgraphs of K_n, satisfying the following properties:

- The rows of A sum to $\lambda = 1$ (without loss of generality).
- $A_{GH} = 0$ whenever $\overline{A \oplus B}$ is non-bipartite.
- The minimal eigenvalue of A is $\lambda_{\min} = -1/7$.

We chose $\lambda_{\min} = -1/7$ since this gives the correct upper bound 1/8 in (1).

How do we go about constructing this matrix? We use symmetries to make our life easier. Suppose A is a symmetric matrix satisfying the above properties (we call such a matrix *a solution*). For a graph G, define a new matrix $A^{\oplus G}_{ST} = A_{S\oplus G, T\oplus T}$. It is not hard to check that $A^{\oplus G}$ is also a solution, and moreover $X = \mathbb{E}_G A^{\oplus G}$ is also a solution, and satisfies $X = X^{\oplus G}$ for all graphs G. This implies that the Fourier characters are the eigenvectors of X:

$$\begin{aligned}
(X\chi_G)_H &= \sum_S X_{HS}\chi_G(S) \\
&= \sum_S X_{\emptyset, S\oplus H}\chi_G(S) \\
&= \sum_S X_{\emptyset, S}\chi_G(S \oplus H) \\
&= \chi_G(H) \sum_S X_{\emptyset, S}\chi_G(S) = \chi_G(H)(X\chi_G)_\emptyset.
\end{aligned}$$

What can the matrix X look like? The subspace of matrices whose eigenvectors are the Fourier characters clearly has dimension $\binom{n}{2}$. It is not hard to check that it is spanned by the matrices B_G for $G \subseteq K_n$, which operate on vectors in $\mathbb{R}^{K_n}$ by $(B_G f)(H) = f(G \oplus H)$. The corresponding eigenvalues are

$$B_G\chi_H = (B_G\chi_H)_\emptyset = \chi_{H\oplus G}(\emptyset) = (-1)^{G\oplus H}.$$

Which of these matrices is feasible? The (G, H) entry of B_S is $e'_G B_S e_H = e'_G e_{H\oplus S}$, and so it is non-zero when $G = H \oplus S$, or in other words when $S = G \oplus H$. For B_S to be feasible, $\overline{S}$ must be bipartite. An inductive argument shows that the space of matrices satisfying the *second* property above (we call such matrices *feasible* is spanned by the matrices B_S for co-bipartite S.

Another symmetry which we can apply is symmetry with respect to renaming of the vertices. Applying this symmetry to the matrix X gives us a matrix which is symmetric with respect to permutations of the vertices.

One can get more constraints by considering the characteristic function f of an optimal family, in our case a triangle-star. A consideration of Hoffman's bound shows that for the bound to be tight, $f - \mathbb{E}f$ must be in the eigenspace of $\lambda_{\min}$. Therefore for any $S \neq \emptyset$ satisfying $\hat{f}(S) \neq 0$, the corresponding eigenvalue must be $\lambda_{\min}$. In other applications of this method (such as Friedgut [14] and Ellis, Friedgut and Pilpel [8]), the space of feasible matrices has small dimension, and these constraints together with $\lambda = 1$ determine the matrix A, and it remains to verify that there are no eigenvalues smaller than the "conjectured" $\lambda_{\min}$. In our case this doesn't happen, and so we have to be more creative. The construction (detailed below) gives a matrix A with the following properties:

1. The eigenvalue corresponding to $\chi_\emptyset$ is $\lambda = 1$.
2. The eigenvalue corresponding to sets χ_G for G a single edge, a pair of edges or a triangle is $\lambda_{\min} = -1/7$.
3. All other eigenvalues are at least $\lambda_2 = -1/7 + 1/952$.

Before explaining the construction, we explain how we can deduce uniqueness and stability, using tools for the analysis of Boolean functions.

5.2 Uniqueness and stability

The matrix A which we have just claimed to exist shows, via (1), that an odd-cycle-agreeing family contains at most $2^{\binom{n}{2}-3}$ graphs. We continue to prove *uniqueness* and *stability*. *Uniqueness* states that the only families attaining this bound are *triangle-semistars*, which are families of the form $\{G \subseteq K_n : G \cap T = S\}$ for some triangle T and $S \subseteq T$ (when $S = T$, this is a *triangle-star*). *Stability* states that the only families containing at least $(1-\epsilon)2^{\binom{n}{2}-3}$ graphs are $O(\epsilon)2^{\binom{n}{2}}$-close to triangle-semistars.

Uniqueness. In view of the reduction of Chung, Frankl, Graham and Shearer, it is enough to prove uniqueness for odd-cycle-intersecting families. The tightness conditions in Hoffman's bound imply that if f is the characteristic function of an odd-cycle-intersecting family $\mathcal{F}$ of size $|\mathcal{F}| = 2^{\binom{n}{2}-3}$ then f is in the span of χ_S for $|S| \leq 3$. A result of Friedgut [14] implies that $\mathcal{F}$ is a 3-star, and so a triangle-star.

Stability. For stability we need to be more careful with our application of Hoffman's bound. Let $\mathcal{F}$ be an odd-cycle-agreeing family of size $|\mathcal{F}| = (1-\epsilon)2^{\binom{n}{2}-3}$, and let f be its characteristic function. Decompose f as $f = f_0 + f_1 + f_2$, where $f_0 = \mathbb{E}[f]\mathbf{1}$, f_1 is in the eigenspace of $-1/7$, and f_2 consists of the rest. Let $\|f_2\|^2 = \tau$. Then

$$0 = \langle f, Af \rangle \geq \lambda \mathbb{E}[f]^2 + \lambda_{\min}(\mathbb{E}[f] - \mathbb{E}[f]^2 - \tau) + \lambda_2 \tau.$$

Arithmetic shows that

$$\tau \leq \frac{-\lambda_{\min}}{\lambda_2 - \lambda_{\min}} \left(\frac{-\lambda_{\min}}{\lambda - \lambda_{\min}} - \mathbb{E}[f] \right) = \frac{\epsilon}{17}.$$

In other words, the Fourier expansion of f is $O(\epsilon)$-close to being supported on the Fourier coefficients of size at most 3. If we replaced 3 with 1, then the Friedgut–Kalai–Naor theorem [15] would imply that f is close to a dictatorship. In this case, the Kindler–Safra theorem [19] states that if ϵ is small enough, f is $O(\epsilon)$-close to a Boolean function g depending on $T = O(1)$ coordinates, that is $\|f - g\|^2 = O(\epsilon)$. (The theorem is false when $T = 3$.)

Let $\mathcal{G}$ be the family corresponding to $\mathcal{F}$. We first claim that $\mathcal{G}$ is odd-cycle-agreeing, if ϵ is small enough. Indeed, suppose not, and take $A, B \in \mathcal{G}$ which are not odd-cycle-agreeing. We can assume that A, B are contained in the set D of T coordinates on which g depends. For each $W \subseteq \overline{D}$, consider the sets $A_W = A \cup W$ and $B_W = B \cup W$. Since $A_W \oplus B_W = A \oplus B$ does not contain an odd cycle, at most one of them can belong to $\mathcal{F}$. Therefore $|\mathcal{F} \oplus \mathcal{G}| \geq 2^{|\overline{D}|}$, and so $\|f - g\|^2 \geq 2^{-T}$. If ϵ is small enough, we reach a contradiction.

We have shown that $\mathcal{G}$ is odd-cycle-agreeing. If $\mathcal{G}$ is a triangle-semistar, then we are done. There are only finitely many possibilities (up to renaming of the vertices) for $\mathcal{G}$, and so if ϵ is small enough, all of them are more than $O(\epsilon)$-far from $\mathcal{F}$. We conclude that $\mathcal{G}$ must be a triangle-semistar.

The compactness argument for stability outlined here follows Friedgut [14] closely. A different argument is used in Ellis, Friedgut and Pilpel [8] to prove stability for t-intersecting families of permutations, though a weak form of stability also follows from Ellis, Filmus and Friedgut [5, 6].

5.3 Constructing the matrix

It remains to construct the matrix A. The idea is to find a large enough collection of feasible matrices which are possible to analyze. Using inclusion-exclusion, it is possible to show that the there for each graph R, there is a feasible matrix Λ_R such that the eigenvalue corresponding to χ_G is $(-1)^{|G|}$ times the probability $q_R(G)$ that $G \cap C \approx R$, where C is a random cut formed by splitting the vertex set into two sets uniformly. In particular, there is a feasible matrix Λ_i such that the eigenvalue corresponding to χ_G is $(-1)^{|G|}$ times the probability $q_i(G)$ that $|G \cap C| = i$.

When G is large (contains many edges), all the probabilities $q_i(G)$ are small. Therefore if we consider a matrix of the form

$$A = \sum_{i=0}^{d} c_i \Lambda_i$$

for some small d, then the eigenvalues corresponding to χ_G will be close to zero for large G. Indeed, the eigenvalue corresponding to χ_G is

$$\lambda_G = (-1)^{|G|} \sum_{i=0}^{d} c_i q_i(G).$$

One could hope that an appropriate choice of the coefficients c_i would then produce the correct eigenvalues.

Consider the following table:

G	$q_0(G)$	$q_1(G)$	$q_2(G)$	$q_3(G)$	$q_4(G)$
$\emptyset$	1	0	0	0	0
$-$	1/2	1/2	0	0	0
$\wedge$	1/4	1/2	1/4	0	0
$\triangle$	1/4	0	3/4	0	0
F_4	1/16	4/16	6/16	4/16	1/16
K_4^-	1/8	0	1/4	1/2	1/8

In the table, F_4 is a forest with 4 edges (they all have the same cut distribution). The first line implies that $c_0 = 1$, so that we get $\lambda = 1$. If f is the characteristic function of a triangle-semistar then $\hat{f}(S) \neq 0$ for all subgraphs of the triangle. For Hoffman's bound to be tight, we need the corresponding eigenvalues to be $\lambda_{\min} = -1/7$. Look at the second and third line, we conclude that $c_1 = -5/7$ and $c_2 = -1/7$. This also works for the fourth line. The following two lines show that $4c_3 + c_4 = 3/7$. This leads us to choose $c_3 = 3/28$ and $c_i = 0$ for $i > 3$.

The idea now is that when $|G|$ is large, a random cut usually cuts more than three edges, and so the eigenvalue corresponding to χ_G is close to zero. It is not so clear what happens when $|G|$ is medium-size, but that can be checked with a computer. Doing that, we obtain the following information concerning

$$A_1 = \Lambda_0 - \frac{5}{7}\Lambda_1 - \frac{1}{7}\Lambda_2 + \frac{3}{7}\Lambda_3 :$$

- The eigenvalue corresponding to $\chi_\emptyset$ is $\lambda = 1$.
- The eigenvalue corresponding to sets χ_G for G a single edge, a pair of edges, a triangle, a quadruple of edges, or a diamond is $\lambda_{\min} = -1/7$.
- All other eigenvalues are at least $\lambda_2(A_1) = -1/7 + 1/56$.

This information is tediously proved in the paper without computer calculations.

The matrix A_1 already gives us the desired upper bound 1/8, but is not quite enough for uniqueness and stability, though in principle these can be recovered by enumerating over all families of graphs over at most 5 edges. Instead of this enumeration, we can "fix" the matrix A_1 by adding to it a matrix A_2 which will get rid of the spurious tight eigenvectors without harming any of the other properties. This matrix is

$$A_2 = \sum_F \Lambda_F - \Lambda_\square,$$

where the sum goes over all forests of size 4. This matrix satisfies the following properties:

- The eigenvalue corresponding to χ_G for $|G| < 4$ is 0.

- The eigenvalue corresponding to forests of size 4 is $1/16$.
- The eigenvalue corresponding to diamonds is $1/8$.
- All other eigenvalues are at most 1 in absolute value.

Using these properties, it is not hard to check that the matrix $A = A_1 + (16/17)A_2$ fits the bill.

References

[1] Rudolf Ahlswede and Levon H. Khachatrian. The complete intersection theorem for systems of finite sets. *Eur. J. Comb.*, 18(2):125–136, 1997.

[2] Rudolf Ahlswede and Levon H. Khachatrian. A pushing-pulling method: New proofs of intersection theorems. *Combinatorica*, 19(1):1–15, 1999.

[3] Fan R. K. Chung, Ronald L. Graham, Peter Frankl, and James B. Shearer. Some intersection theorems for ordered sets and graphs. *J. Comb. Theory, Ser. A*, 43(1):23–37, 1986.

[4] Michel Deza and Péter Frankl. On the maximum number of permutations with given maximal or minimal distance. *J. Comb. Theory A*, 22:352–360, 1977.

[5] David Ellis, Yuval Filmus, and Ehud Friedgut. A quasi-stability result for dictatorships in S_n. *Submitted.*

[6] David Ellis, Yuval Filmus, and Ehud Friedgut. A quasi-stability result for low-degree Boolean functions on S_n. *Preprint.*

[7] David Ellis, Yuval Filmus, and Ehud Friedgut. Triangle-intersecting families of graphs. *J. Eur. Math. Soc.*, 14(3):841–885, 2012.

[8] David Ellis, Ehud Friedgut, and Haran Pilpel. Intersecting families of permutations. *J. Am. Math. Soc.*, 24(3):649–682, 2011.

[9] Paul Erdős, Chao Ko, and Richard Rado. Intersection theorems for systems of finite sets. *Quart. J. Math. Oxford (2)*, 12:313–320, 1961.

[10] Péter Frankl. On Sperner families satisfying an additional condition. *J. Comb. Th. A*, 20:1–11, 1976.

[11] Péter Frankl. Erdős-Ko-Rado theorem with conditions on the maximal degree. *J. Comb. Th. A*, 46:252–263, 1987.

[12] Péter Frankl and Zoltán Füredi. Non-trivial intersecting families. *J. Comb. Th. A*, 41:150–153, 1986.

[13] Péter Frankl and Richard M. Wilson. The Erdös–Ko–Rado theorem for vector spaces. *J. Comb. Th. A*, 43(2):228–236, 1986.

[14] Ehud Friedgut. On the measure of intersecting families, uniqueness and stability. *Combinatorica*, 28(5):503–528, 2008.

[15] Ehud Friedgut, Gil Kalai, and Assaf Naor. Boolean functions whose Fourier transform is concentrated on the first two levels. *Adv. App. Math.*, 29(3):427–437, 2002.

[16] Anthony J. W. Hilton and Eric C. Milner. Some intersection theorems for systems of finite sets. *Quart. J. Math. Oxford (2)*, 18:369–384, 1967.

[17] Alan J. Hoffman. On eigenvalues and colorings of graphs, I. In *Graph Theory and its Applications*, Proc. Advanced Sem., Math. Research Center, Univ. of Wisconsin, Madison, Wis., pages 79–91. Academic Press, New York, 1969.

[18] Gyula O. H. Katona. A simple proof of the Erdős-Chao Ko-Rado theorem. *J. Comb. Th. B*, 13:183–184, 1972.

[19] Guy Kindler. *Property testing, PCP and Juntas*. PhD thesis, Tel-Aviv University, 2002.

[20] László Lovász. On the Shannon capacity of a graph. *IEEE Trans. Inform. Theory*, 25:1–7, 1979.

[21] L. Pyber. A new generalization of the Erdős-Ko-Rado theorem. *J. Comb. Th. A*, 43:85–90, 1986.

[22] Richard M. Wilson. The exact bound in the Erdős-Ko-Rado theorem. *Combinatorica*, 4:247–257, 1984.