

# Power Distribution in Randomized Weighted Voting: the Effects of the Quota

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We study the Shapley value in weighted voting games. The Shapley value has been used as an index for measuring the power of individual agents in decision-making bodies and political organizations, where decisions are made by a majority vote process. We characterize the impact of changing the quota (i.e., the minimum number of seats in the parliament that are required to form a coalition) on the Shapley values of the agents. Contrary to previous studies, which assumed that the agent weights (corresponding to the size of a caucus or a political party) are fixed, we analyze new domains in which the weights are stochastically generated, modeling, for example, election processes.

We examine a natural weight generation process: the Balls and Bins model, with uniform as well as exponentially decaying probabilities. We also analyze weights that admit a super-increasing sequence, answering several open questions pertaining to the Shapley values in such games.

Categories and Subject Descriptors: J.4 [Social and Behavioral Sciences]: Sociology

## 1. INTRODUCTION

Weighted voting is a common method for making group decisions. This is the method used in parliaments: one can think of the political parties in a parliament as *weighted agents*, where an agent's weight is the number of seats it holds in the parliament.

Power dynamics in electoral systems have been the focus of academic study for several decades. One important observation is that the weight of a party is not necessarily equal to its electoral power. For example, consider a parliament that has three parties, two with 50 seats, and one with 20 seats. Assuming that a majority of the votes is required in order to pass a bill, all three parties have the same decision-making power: no single party can pass a bill on its own, whereas any two parties can. This contrasts the fact that one of the parties has significantly less weight than the other two. One of the most prominent measures of voting power is the Shapley–Shubik power index (also referred to as the Shapley value); it has played a central role in the analysis of real-life voting systems, such as the US electoral college [Mann and Shapley 1960, 1962], the EU council of members [Leech 2002; Słomczyński and Życzkowski 2006; Felsenthal and Machover 1998], and the UN security council [Shapley 1953].

Empirical studies of weighted voting present an interesting phenomenon: changes to the quota (i.e., the number of votes required in order to pass a bill, also called the threshold) can dramatically affect agent voting power. Changes to the quota have been proposed as a way to correct power imbalance in the EU council of members [Słomczyński and Życzkowski 2006]; this is because quota changes are perceived as a preferable alternative to changes to agent weights (as is proposed by Penrose [1946]), and were thus argued for in [Słomczyński and Życzkowski 2006; Leech and Machover 2003].

The objective of this paper is to study the effects of changes to the quota on electoral power, as measured by the Shapley–Shubik power index. Previous analytical studies of power indices as a function of the quota have mostly focused on the following question: given a set of weights, what would be the effect of changes to the quota on voting power? Previous studies have focused on the analysis of fixed, arbitrary weight vectors, with limited success.

Instead of studying arbitrary weight vectors, we assume that weights are sampled from certain natural distributions. Studying weights that are sampled from distributions carries two benefits. First, it allows us to reason more deeply on the effects that quota changes have on voting power; second, randomly sampled weights can have a natural interpretation, especially in electoral settings. Weights sampled from a prescribed distribution can naturally be thought of as modeling a parliamentary

election process that determines the number of seats each party will hold. Using natural weight generation processes, we analyze the expected behavior of the Shapley value as a function of the quota. Our results show that voting power can behave in a rather unusual manner; for example, some of our results show that even when weights are likely to be very similar, some choices of a quota will cause significant differences in voting power. Understanding the behavior of voting power when the quota changes is important, as changing the quota is a simple method of controlling agents' electoral influence. For example, suppose that a legislative body wants to ensure that electoral power satisfies a certain property (say, that all parties have approximately the same voting power, or that power disparity is high). If one assumes a certain probabilistic model of how weights are obtained, our results ensure that such properties are in effect (with high probability).

### 1.1. Our Contributions

Our work focuses on the *Balls and Bins* model—a model that has received considerable recent attention in the computer science community [Mitzenmacher et al. 2000; Mitzenmacher and Upfal 2005; Raab and Steger 1998]. Informally, in this iterative process, in each round a ball is thrown into one of several bins according to a fixed probability distribution. The process takes place in  $m$  rounds, and at the end of the process, the weight of agent  $i$  is given by the number of balls in its corresponding bin  $i$ .

In Section 4, we study a simple model, where each ball lands in one of the  $n$  bins uniformly at random. We identify a repetitive fluctuation pattern in the Shapley values, with cycles of length  $\frac{m}{n}$ . We show that if the quota is sufficiently bounded away from the borders of its length- $\frac{m}{n}$  cycle, then the Shapley values of all agents are likely to be very close to each other. On the other hand, we show that due to noise effects, when the quota is situated close enough to small multiples of  $\frac{m}{n}$ , the highest Shapley value can be roughly double than that of the smallest one. In other words, even if one expects that candidate weights are identical with high probability, choosing a quota near a multiple of  $\frac{m}{n}$  may result in a great difference between Shapley values.

To complement our findings for the uniform case, in Section 5 we consider the case in which the probabilities decay exponentially, with a decay factor no larger than  $1/2$ . We show that analyzing this case essentially boils down to characterizing the Shapley values in a game where weights are a super-increasing sequence (Section 6). Our results significantly strengthen previous results obtained for this case by Zuckerman et al. [2012]: we fully characterize the Shapley value as a function of the quota for the super-increasing case. In addition to giving a closed-form formula for the Shapley values, we also provide conditions for the equality of consecutive agents.

### 1.2. Related Work

Weighted voting games (WVGs) have been studied extensively, two classic power measures proposed by Banzhaf [1964] and by Shapley [1953] being the main object of analysis (see Felsenthal and Machover [1998], Maschler et al. [2013] and Peleg and Sudhölter [2007] for expositions). From an economic point of view, the appeal of the Shapley value is that it is the *only* division rule that satisfies certain desirable axioms [Shapley 1953; Shapley and Shubik 1954]. Computing Shapley values in WVGs has also been the focus of several studies: power indices have been shown to be computationally intractable (see Chalkiadakis et al. [2011] for a detailed overview), but easily approximable. Randomized sampling has been employed in order to efficiently approximate the Shapley value; the earliest example of this technique being used appears in [Mann and Shapley 1960], with subsequent analysis by Bachrach et al. [2010] and Fatima et al. [2008]. However, this type of analysis employs the inherent probabilistic nature of power indices, rather than inducing randomness in the weighted voting game itself.

If one makes no assumptions on weight distributions, very little can be said about the effects of the quota on WVGs. Indeed, as demonstrated in [Zick et al. 2011; Zick 2013; Zuckerman et al. 2012], power measures are highly sensitive to varying quota values. While Zick [2013] presents some preliminary results on the effects of the quota when weights are sampled from a given distribution, our work takes a more principled approach to the matter.

Several works have studied the effects of randomization on weighted voting games from a theoretical, computational, and empirical perspective. The earliest study of randomization and its effects on voting power is due to Penrose [1946], who shows that the Banzhaf power index scales as the square root of agents' weight when weights are drawn from bounded distributions.<sup>1</sup> Lindner [2004] shows certain convergence results for power indices, when agents are sampled from some distributions; Tauman and Jelnov [2012] show that when weights are sampled from the uniform distribution, the expected Shapley value of an agent is proportional to its weight, assuming that the quota is at least 50%. Zick [2013] considers a model where the quota is sampled from a uniform distribution, and bounds the variance of the Shapley value in this setting, both for general weights and for weights sampled from certain distributions. Finally, the effects of changes to the quota have also been studied empirically, mostly in the context of the EU council of members [Leech and Machover 2003; Leech 2002; Słomczyński and Życzkowski 2006].

## 2. PRELIMINARIES

*General notation.* Given a vector  $x \in \mathbb{R}^n$  and a set  $S \subseteq \{1, \dots, n\}$ , let  $x(S) = \sum_{i \in S} x_i$ . For a random variable  $X$ , we let  $\mathbb{E}[X]$  be its expectation, and  $\text{Var}[X]$  be its variance. For a set  $S$ , we denote by  $\binom{S}{k}$  the collection of subsets of  $S$  of cardinality  $k$ . The notation  $T \in_R \binom{S}{k}$  means that the set  $T$  is chosen uniformly at random from  $\binom{S}{k}$ . We let  $B(n, p)$  denote the binomial distribution with  $n$  trials and success probability  $p$ . We let  $\mathcal{N}(\mu, \sigma^2)$  denote the normal distribution with mean  $\mu$  and variance  $\sigma^2$ . We let  $U(a, b)$  denote the uniform distribution on the interval  $[a, b]$ .

We let  $O_p(\cdot)$  denote the usual big-O notation, conditioned on a fixed value of  $p$ . In other words, having  $f(n) = O_p(g(n))$  means that there exist functions  $K(\cdot), N(\cdot)$ , such that for  $n \geq N(p)$ ,  $f(n) \leq K(p) \cdot g(n)$ .

Finally, for a distribution  $D$  over  $\mathbb{R}$  and some event  $\mathcal{E}$ , we simplify our notation by letting  $\Pr[\mathcal{E}(D)] = \Pr_{x \sim D}[\mathcal{E}(x)]$ . For example, for  $a > 0$ , we can write  $\Pr[B(n, p) \leq a] = \Pr_{x \sim B(n, p)}[x \leq a]$ .

*Weighted voting games.* A *weighted voting game* (WVG) is given by a set of agents  $N = \{1, \dots, n\}$ , where each agent  $i \in N$  has a non-negative weight  $w_i$ , and a *quota* (or *threshold*)  $q$ . Unless otherwise specified, we assume that the weights are arranged in non-decreasing order,  $w_1 \leq \dots \leq w_n$ . For a subset of agents  $S \subseteq N$ , we define  $w(S) = \sum_{i \in S} w_i$ .

A subset of agents  $S \subseteq N$  is called *winning* (has value 1) if  $w(S) \geq q$  and is called *losing* (has value 0) otherwise.

*The Shapley value.* Let  $\text{Sym}_n$  be the set of all permutations of  $N$ . Given some permutation  $\sigma \in \text{Sym}_n$  and an agent  $i \in N$ , we let  $P_i(\sigma) = \{j \in N : \sigma(j) < \sigma(i)\}$ ;  $P_i(\sigma)$  is called the set of  $i$ 's predecessors in  $\sigma$ . Let us write  $m_i(S)$  to be  $v(S \cup \{i\}) - v(S)$ ; in other words,  $m_i(S) = 1$  if and only if  $v(S) = 0$  but  $v(S \cup \{i\}) = 1$ . If  $m_i(S) = 1$ , we say that  $i$  is *pivotal* for  $S$ ; similarly, we write  $m_i(\sigma) = m_i(P_i(\sigma))$ , and say that  $i$  is pivotal for  $\sigma \in \text{Sym}_n$  if  $i$  is pivotal for  $P_i(\sigma)$ . The Shapley–Shubik power index (often referred to as the Shapley value in the context of WVG's) is simply the probability that  $i$  is pivotal for a permutation  $\sigma \in \text{Sym}_n$  selected uniformly at random. More explicitly,

$$\varphi_i = \frac{1}{n!} \sum_{\sigma \in \text{Sym}_n} m_i(\sigma).$$

<sup>1</sup>The results shown by Penrose predate the work by Banzhaf, but can be applied directly to his work; see [Felsenthal and Machover 2005] for details.

Since  $\sigma^{-1}(i)$  is distributed uniformly when  $\sigma$  is chosen at random from  $\text{Sym}_n$ , we also have the alternative formula

$$\varphi_i = \frac{1}{n} \sum_{\ell=0}^{n-1} \mathbb{E}_{S \in \mathcal{R}[\binom{N \setminus \{i\}}{\ell}]} m_i(S). \quad (1)$$

*Properties of the Shapley value.* For WVG's, it is not hard to show that  $w_i \leq w_j$  implies  $\varphi_i \leq \varphi_j$ , and so if the weights are arranged in non-decreasing order, the minimal Shapley value is  $\varphi_1$  and the maximal one is  $\varphi_n$ . Another useful property that follows immediately from the definitions is that  $\sum_{i \in N} \varphi_i = 1$ , assuming  $0 < q \leq \sum_{i \in N} w_i$ . When we want to emphasize the role of the quota  $q$ , we will think of the Shapley values as functions of  $q$ :  $\varphi_i(q)$ .

### 3. AN OVERVIEW OF OUR RESULTS

We begin by briefly presenting our three major contributions.

*The Balls and Bins Distribution: the Uniform Case.* In this work, we study the effects of the quota on agents' voting power, when agent weights are sampled from the balls and bins distribution. This distribution is appealing, as it can naturally model election dynamics under plurality voting: consider an election where  $m$  voters vote for  $n$  parties; the weight of each party is determined by the number of votes it receives. If we assume that each voter will vote for party  $i$  with probability  $p_i$ , party seats are distributed according to the balls and bins distribution with the probability vector  $\mathbf{p}$ . We first study the case where balls are thrown into bins uniformly at random; that is, voters choose parties uniformly at random (the *impartial culture* assumption).

When weights are drawn from a uniform balls and bins distribution with  $m$  balls and  $n$  bins, Shapley values follow a rather curious fluctuation pattern as the quota varies (see Figure 1). Note that the fluctuation is quite regular, with power disparity occurring at regular intervals (these intervals are of length  $\frac{m}{n}$ ). Our first result (Theorem 4.2) shows that when we select a quota that is sufficiently far from an integer multiple of  $\frac{m}{n}$ , all agents' Shapley values tend to be the same. When the quota is an integer multiple of  $\frac{m}{n}$ , we distinguish between two cases; when the quota is far from the 50% mark, power disparity is likely to occur, with the weakest agent's voting power sinking to less than half that of the strongest (Theorem 4.3). However, disparity is mitigated when the quota is near the 50% mark (Theorem 4.4). These results indicate that even if weights are likely to be similar (as is the case for the uniform balls and bins distribution), power disparity is likely in certain quotas.

*The Balls and Bins Distribution: the Exponential Case.* In Section 5, we explore the case where the voting probabilities are exponentially increasing, i.e.  $\frac{p_i}{p_{i+1}} = \rho$  for some fixed constant  $0 < \rho < \frac{1}{2}$ . In this case, we show (Theorem 5.5) that agents' weights are very likely to be *super-increasing* (super-increasing weights were first studied by Zuckerman et al. [2012]). Thus, in order to understand the expected behavior of voting power as a function of the quota in the exponential setting, it suffices to characterize the Shapley value for weighted voting games with super-increasing weights.

*Super-Increasing Weights.* Following the crucial observation made in Theorem 5.5, we completely characterize voting power in WVG's with super-increasing weights in Section 6. First we show that in order to compute the Shapley value of an agent under a super-increasing sequence, it suffices to know her Shapley value when weights are powers of 2 (Lemma 6.2). This connection leads to a closed-form formula for the Shapley value when weights are super-increasing (Theorem 6.3). Employing our formula, we are able to derive some interesting properties of the Shapley values as functions of the quota for super-increasing sequences. These results generalize those found in [Zuckerman et al. 2012], providing a clear understanding of the mechanics of power distribution and the quota for the case of super-increasing sequences.

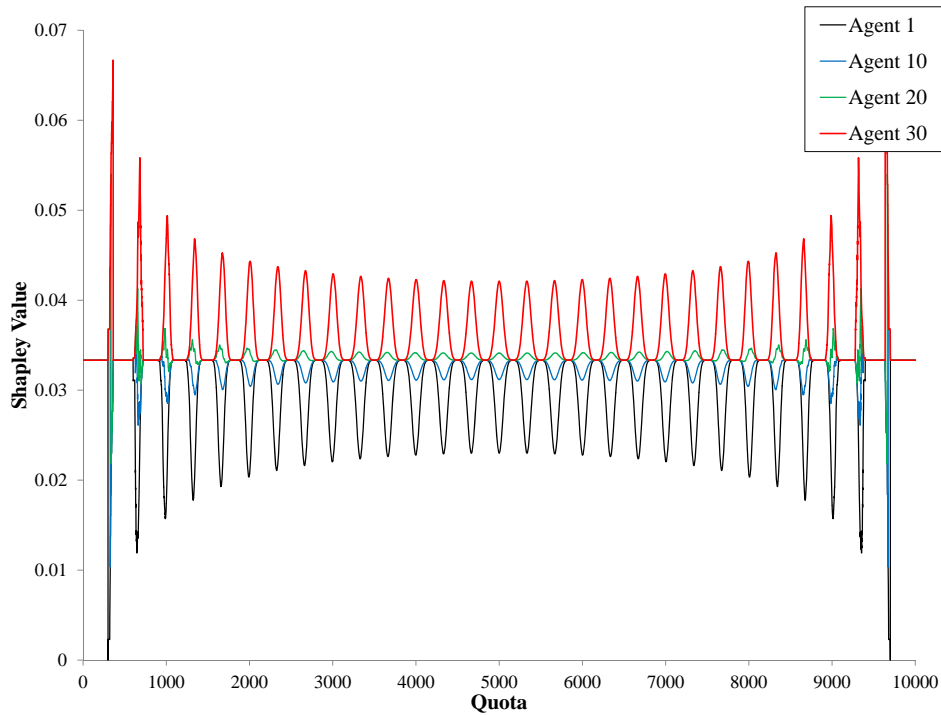


Fig. 1: The Shapley values of agents 1, 10, 20 and 30 in a 30-agent WVG where weights were drawn from a balls and bins distribution with  $m = 10000$  balls.

We conclude our study with an interesting analysis of  $\varphi_i(q)$  when weights are finite prefixes of the sequence  $(2^m)_{m=0}^\infty$ . The analysis explains in many ways the fractal shape of  $\varphi_i(q)$  when weights are powers of two, and shows when voting power will increase or decrease.

#### 4. THE BALLS AND BINS DISTRIBUTION: THE UNIFORM CASE

We consider a generative stochastic process called the Balls and Bins process. In its most general form, given a set of  $n$  bins and a distribution represented by a vector  $\mathbf{p} \in [0, 1]^n$  such that  $\sum_{i=1}^n p_i = 1$ , the process unfolds in  $m$  steps. At each step, a ball is thrown into one of the bins based on the probability vector  $\mathbf{p}$ . The resulting weights are then sorted in non-decreasing order  $w_1 \leq \dots \leq w_n$ .

We begin our study of the balls and bins process by considering the most commonly studied version of the balls and bins model, in which each ball is thrown into one of the bins with equal probability, i.e.,  $p_i = 1/n$ , for all  $i \in N$ .

As Figure 1 shows for the case of  $n = 30$ , the behavior of the Shapley values demonstrates an almost perfect cyclic pattern, with intervals of length  $m/n$ . As can be seen in the figure, for quota values that are sufficiently distant from the interval endpoints, all of the Shapley values tend to be equal (as the Shapley values of the highest and lowest agents are equal in these regions). As the number of balls grows, all of the bins tend to have nearly the same number of balls in them; however, low weight discrepancy does not immediately translate to low power discrepancy: we can guarantee nearly equal voting power in some quotas, but not in others.

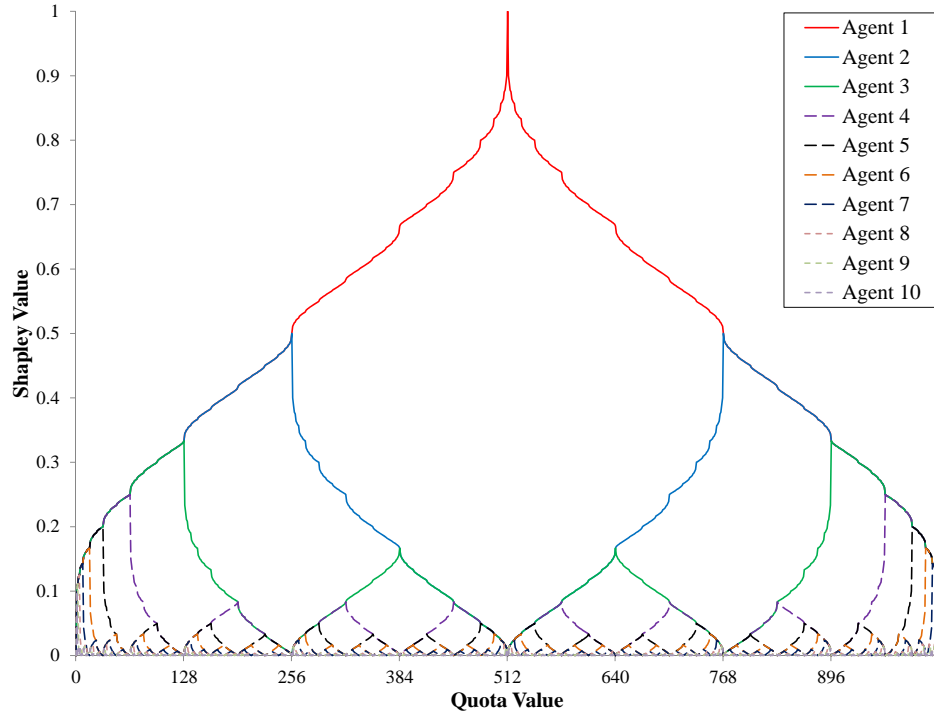


Fig. 2: The Shapley values as a function of the quota in a 10-agent game where agent  $i$ 's weight is  $2^{n-i}$ .

From a legislative perspective, our theorems provide some interesting guarantees on voting power, under the assumption that voter behavior follows a uniform Balls and Bins distribution. If one wishes to ensure that all parties have similar voting power, it suffices to set a quota that is sufficiently far from  $\frac{m}{n}$  (and is close enough to 50%); if one wishes to ensure power disparity, setting a quota closer to  $\frac{m}{n}$  is desirable.

We begin by providing a formula for the differences between two Shapley values.

LEMMA 4.1. *For all agents  $i, j \in N$ ,*

$$|\varphi_j - \varphi_i| = \frac{1}{n-1} \sum_{\ell=0}^{n-2} \Pr_{S \in R[\binom{N \setminus \{i,j\}}{\ell}]} [q - \max(w_i, w_j) \leq w(S) < q - \min(w_i, w_j)].$$

We now give a theoretical justification for the near-identity of Shapley values for quotas that are well away from integer multiples of  $\frac{m}{n}$ .

THEOREM 4.2. *Let  $M = \frac{m}{3n^3}$ . Suppose that  $|q - \frac{\ell m}{n}| > \frac{1}{\sqrt{M}} \frac{m}{n}$  for all integers  $\ell$ . Then with probability  $1 - 2(\frac{2}{e})^n$ , all Shapley values are equal to  $1/n$ .*

The idea of the proof is the following. Suppose that  $w_i \leq w_j$ . According to Lemma 4.1,  $\varphi_i \neq \varphi_j$  only if for some  $S \subseteq N \setminus \{i, j\}$ , we have  $q - w_j \leq w(S) < q - w_i$ . For a fixed set of agents  $|S| \in \binom{N \setminus \{i,j\}}{k}$  we have  $S \sim B(m, k/n)$  — as each ball enters into one of the bins corresponding

to  $S$  with probability  $k/n$ . As a result,  $w(S)$  is concentrated around the mean  $km/n$ . On the other hand  $q - w_j, q - w_i \approx q - \frac{m}{n}$ . Therefore, if  $q$  is far away from  $\frac{(k+1)m}{n}$  for all  $0 \leq k \leq n-2$ , then the probability that  $q - w_j \leq w(S) < q - w_i$  is very small. The details can be found in the appendix.

Returning to our voting setting, the interpretation of Theorem 4.2 is that if the voter population is much larger than the number of candidates, and the votes are assumed to be cast uniformly at random (i.e., a totally neutral distribution of preferences), then choosing a quota that is well away from a multiple of  $\frac{m}{n}$  will most probably lead to an even distribution of power among the elected representatives (e.g., political parties).

#### 4.1. How Weak Can the Weakest Agent Get in the Uniform Case?

As Theorem 4.2 demonstrates, if the quota is sufficiently bounded away from any integral multiple of  $\frac{m}{n}$ , then the distribution of power tends to be even among the agents. When the quota is close to an integer multiple of  $\frac{m}{n}$ , it may very well be that the resulting weighted voting game may not display such an even distribution of power, as a result of weight differences, which originate in the intrinsic “noise” involved in the process. Figure 1 provides an empirical validation of this intuition. Motivated by these observations, we now proceed to study the expected Shapley value of the weakest agent,  $\varphi_1$  (recall that we assume that the weights are given in non-decreasing order).

We present two contrasting results. Let  $q = \ell \cdot \frac{m}{n}$ , for an integer  $\ell$ . When  $\ell = o(\log n)$ , we show that the expected minimal Shapley value is roughly  $\frac{1}{2n}$ , and so it is at least half the maximal Shapley value (in expectation).

**THEOREM 4.3.** *Let  $q = \ell \cdot \frac{m}{n}$  for some integer  $\ell = o(\log n)$ . For  $m = \Omega(n^3 \log n)$ ,  $\mathbb{E}[\varphi_1] = \frac{1}{2n} \pm o(\frac{1}{n})$ .*

In contrast, when  $\ell = \Omega(n)$ , this effect disappears.

**THEOREM 4.4.** *Let  $q = \ell \cdot \frac{m}{n}$  for  $\ell \in \{1, \dots, n\}$  such that  $\gamma \leq \frac{\ell}{n} \leq 1 - \gamma$  for some constant  $\gamma > 0$ . Then for  $m = \Omega(n^3)$ ,  $\mathbb{E}[\varphi_1] \geq \frac{1}{n} - O_\gamma \left( \sqrt{\frac{\log n}{n^3}} \right)$ .*

The idea behind the proof of both theorems is the formula for  $\varphi_1$  given in Lemma 4.5. In this formula and in the rest of the section, the probabilities are taken over both the displayed variables and the choice of weights.

**LEMMA 4.5.** *Let  $q = \ell \cdot \frac{m}{n}$ , where  $\ell \in \{1, \dots, n-1\}$ . For  $m = \Omega(n^3 \log n)$ ,*

$$\mathbb{E}[\varphi_1] = \frac{1}{2(n-\ell)} - \frac{\ell}{n(n-\ell)} + \frac{1}{n-\ell} \Pr_{A \in \mathcal{R} \left[ \begin{smallmatrix} N \setminus \{1\} \\ \ell-1 \end{smallmatrix} \right]} [w(A) + w_1 \geq q] \pm O \left( \frac{1}{n^2} \right).$$

The rather involved proof appears in the appendix.

In order to estimate the expression  $\Pr_{A \in \mathcal{R} \left[ \begin{smallmatrix} N \setminus \{1\} \\ \ell-1 \end{smallmatrix} \right]} [w(A) + w_1 \geq q]$ , we need a good estimate for  $w_1$ . Such an estimate is given by the following lemma.

**LEMMA 4.6.** *With probability  $1 - 2/n$ , we have that  $\sqrt{\frac{m \log n}{3n}} \leq \frac{m}{n} - w_1 \leq \sqrt{\frac{4m \log n}{n}}$ .*

We obtain this bound by applying the *Poisson approximation technique* to the Balls and Bins process, which we now roughly describe. Consider the case of a random event, defined with respect to the weight distribution induced by the process. The probability of the event can be well-approximated by the probability of an analogous event, defined with respect to  $n$  *i.i.d.* Poisson random variables, assuming the event is monotone in the number of balls.

We can now prove Theorem 4.3.

PROOF OF THEOREM 4.3. Lemma A.5 (a simple technical result proved in the appendix) shows that

$$\Pr_{A \in \mathcal{R}[\binom{N \setminus \{1\}}{\ell-1}]} [w(A) + w_1 \geq q] \leq \frac{n}{n - \ell + 1} \Pr[\mathbf{B}(m, \frac{\ell-1}{n}) \geq q - w_1].$$

The concentration bound on  $w_1$  (Lemma 4.6) shows that with probability  $1 - 2/n$ ,  $q - w_1 \geq \frac{(\ell-1)m}{n} + \sqrt{\frac{m \log n}{3n}}$ . Assuming this, a Chernoff bound gives

$$\Pr[\mathbf{B}(m, \frac{\ell-1}{n}) \geq q - w_1] \leq \Pr[\mathbf{B}(m, \frac{\ell-1}{n}) \geq \frac{(\ell-1)m}{n} + \sqrt{\frac{m \log n}{3n}}] \leq e^{-\frac{m \log n / (3n)}{3(\ell-1)m/n}} = o(1),$$

using  $\ell = o(\log n)$ . Accounting for possible failure of the bound on  $q - w_1$ , we obtain

$$\Pr_{A \in \mathcal{R}[\binom{N \setminus \{1\}}{\ell-1}]} [w(A) + w_1 \geq q] \leq \left(1 - \frac{2}{n}\right) \cdot o\left(\frac{n}{n - \ell}\right) + \frac{2}{n} \cdot 1 = o(1),$$

using  $\ell = o(\log n)$ . Lemma 4.5 therefore shows that

$$\mathbb{E}[\varphi_1] \leq \frac{1}{2(n - \ell)} + o\left(\frac{1}{n - \ell}\right) + O\left(\frac{1}{n^2}\right) = \frac{1}{2n} + o\left(\frac{1}{n}\right),$$

since  $\ell = o(\log n)$  implies  $\frac{1}{n - \ell} = \frac{1}{n} + \frac{\ell}{n(n - \ell)} = \frac{1}{n} + o\left(\frac{1}{n}\right)$ . Lemma 4.5 also implies a matching lower bound:

$$\mathbb{E}[\varphi_1] \geq \frac{1}{2(n - \ell)} - \frac{\ell}{n(n - \ell)} - O\left(\frac{1}{n^2}\right) \geq \frac{1}{2n} - o\left(\frac{1}{n}\right).$$

□

In the regime of  $\ell$  addressed by Theorem 4.3,  $\Pr_{A \in \mathcal{R}[\binom{N \setminus \{1\}}{\ell-1}]} [w(A) + w_1 \geq q]$  was negligible. In contrast, in the regime of  $\ell$  addressed by Theorem 4.4,  $\Pr_{A \in \mathcal{R}[\binom{N \setminus \{1\}}{\ell-1}]} [w(A) + w_1 \geq q] \approx 1/2$ , as the following lemma, which is proved in the appendix using the Berry–Esseen theorem, shows.

LEMMA 4.7. *Suppose  $q = \ell \frac{m}{n}$  for an integer  $\ell$  satisfying  $\gamma \leq \frac{\ell-1}{n} \leq 1 - \gamma$ , and let*

$$t_\varepsilon = \Pr_{A \in \mathcal{R}[\binom{N \setminus \{1\}}{\ell-1}]} \left[ w(A) + w_1 \geq q : w_1 = \frac{m}{n} - \varepsilon \sqrt{\frac{m \log n}{n}} \right].$$

Then for  $m \geq 4n^3$ ,

$$t_\varepsilon \geq \frac{1}{2} - \frac{\varepsilon}{2\pi\gamma} \sqrt{\frac{\log n}{n}} - \frac{1}{n}.$$

As Lemma 4.6 shows,  $1/3 \leq \varepsilon \leq 4$  with probability  $1 - 2/n$ , which explains the usefulness of this bound. We can now prove Theorem 4.4.

PROOF OF THEOREM 4.4. Lemma 4.6 shows that with probability  $1 - 2/n$ ,  $w_1 = \frac{m}{n} - \varepsilon \sqrt{\frac{m \log n}{n}}$  for some  $1/3 \leq \varepsilon \leq 4$ , in which regime Lemma 4.7 shows that  $t_\varepsilon \geq \frac{1}{2} - \frac{2}{\pi\gamma} \sqrt{\frac{\log n}{n}} - \frac{1}{n}$ . Accounting for the case in which  $\varepsilon$  is out of bounds,

$$\Pr_{A \in \mathcal{R}[\binom{N \setminus \{1\}}{\ell-1}]} [w(A) + w_1 \geq q] \geq \left(1 - \frac{2}{n}\right) \left(\frac{1}{2} - \frac{2}{\pi\gamma} \sqrt{\frac{\log n}{n}} - \frac{1}{n}\right) \geq \frac{1}{2} - \frac{2}{\pi\gamma} \sqrt{\frac{\log n}{n}} - \frac{3}{n}.$$



Substituting this in Lemma 4.5, we obtain

$$\begin{aligned}\mathbb{E}[\varphi_1] &\geq \frac{1}{2(n-\ell)} - \frac{\ell}{n(n-\ell)} + \frac{1}{n-\ell} \left( \frac{1}{2} - \frac{2}{\pi\gamma} \sqrt{\frac{\log n}{n}} - \frac{3}{n} \right) - O\left(\frac{1}{n^2}\right) \\ &= \frac{1}{n-\ell} - \frac{\ell}{n(n-\ell)} - \frac{1}{n-\ell} O_\gamma\left(\sqrt{\frac{\log n}{n}}\right) - O\left(\frac{1}{n^2}\right) = \frac{1}{n} - O_\gamma\left(\sqrt{\frac{\log n}{n^3}}\right).\end{aligned}$$

□

## 5. THE BALLS AND BINS DISTRIBUTION: THE EXPONENTIAL CASE

In Section 4, we showed that even when the distribution is not inherently biased towards any agent, substantial inequalities may arise due to random noise. We now turn to study the case in which the distribution is strongly biased. Returning to our formal definition of the general balls and bins process, we assume that the probabilities in the vector  $\mathbf{p}$  are ordered in increasing order and  $\frac{p_i}{p_{i+1}} = \rho$ , for some  $\rho < 1/2$ . We observe that as  $m$  approaches  $\infty$ , the weight vector follows a power law with probability 1, where for each  $i = 1, \dots, n-1$ ,  $\frac{w_i}{w_{i+1}} = \rho$ . A closely related family of weight vectors that we will refer to is the family of *super-increasing* weight vectors:

*Definition 5.1 (Zuckerman et al. [2012]).* A series of positive weights  $\mathbf{w} = (w_1, \dots, w_n)$  is said to be super-increasing (SI) if for every  $i = 1, \dots, n$ ,  $\sum_{j=1}^{i-1} w_j < w_i$ .

The following three results (Lemma 5.2, Lemma 5.4 and Theorem 5.5) show that for a sufficiently large value of  $m$ , estimating the Shapley values in WVG's where the weights are sampled from an exponential distribution can be reduced to the study of Shapley values in a game with a prescribed (fixed) SI weight vector. This insight is useful, as voting games with SI weights are much easier to analyze; Section 6 completely characterizes the behavior of voting power in WVG's with SI weights.

The following lemma characterizes the necessary size of the voter population, so as to make the weight vector super-increasing, if the voters vote according to the above exponential distribution. The following lemma shows that when the weights are sampled from an exponential distribution, given enough voters, it is highly probable that the resulting weights are super-increasing.

**LEMMA 5.2.** *Assume that  $m$  voters submit the votes according to the exponential distribution over candidates in which the probability the candidate  $i$  is voted for by each voter is proportional to  $\rho^{-i}$ , for some  $0 < \rho < 1/2$ . There is a (universal) constant  $C > 0$  such that if  $m \geq C\rho^{-n}(1-2\rho)^{-2} \log n$  then the resulting weight vector is super-increasing with probability  $1 - O(\frac{1}{n})$ . Furthermore, as  $m \rightarrow \infty$ , the probability approaches 1.*

The proof of the lemma uses a standard concentration bound (see Appendix B).

Before we proceed, it would be helpful to provide some intuition about the behavior of the Shapley values. Assuming that agent weights are given by an increasing sequence  $\mathbf{w}$  of  $n$  reals, consider the set of all distinct subset sums of the weights  $\mathcal{S}(\mathbf{w}) = \{s : \exists P \subseteq [n] \text{ s.t. } s = \sum_{i \in P} w_{n+1-i}\}$  (we use  $w_{n+1-i}$  instead of  $w_i$  to make some formulas below nicer). Furthermore, suppose that the subset sums are ordered in increasing order; i.e.,  $\mathcal{S}(\mathbf{w}) = \{s_j\}_{j=1}^t$ , such that  $s_j < s_{j+1}$  for  $1 \leq j < t$ . It is easy to show, using the definition of the Shapley value, that for any quota  $q \in (s_j, s_{j+1}]$ , for  $1 \leq j < t$ , the Shapley values of every agent  $i \in N$  remain constant at some value  $\varphi_i(j)$ , defined for the  $j$ 'th interval. We formalize this intuition in Section 6, where we give a formula for  $\varphi_i(j)$ .

Before we state the formula (Proposition 5.3 below), we need some notation. For each  $P \subseteq N$ , let  $\tilde{w}(P) = \sum_{i \in P} w_{n+1-i}$ . For some  $j$ ,  $\tilde{w}(P) = s_j$ , where  $s_j \in \mathcal{S}$ . If  $P \neq N$  then  $j < t$  and so  $s_{j+1} = \tilde{w}(P^+)$  for some  $P^+ \subseteq N$ . Write  $I_P^w = (\tilde{w}(P), \tilde{w}(P^+)]$ . Then by definition, the intervals  $I_P^w$  partition the interval  $(0, w(N)]$ . We can now state the formula for  $\varphi_i(j)$ . Given a weight vector

$w$ , let  $\varphi_i^w(q)$  denote the Shapley value of agent  $i$  when the quota is  $q$  and the weights are given by  $w$ .

**PROPOSITION 5.3.** *Suppose that  $w = (w_1, \dots, w_n)$  is a SI sequence of weights, and suppose that  $q \in (0, w(N)]$ , say  $q \in I_P^w$  for some  $P \subseteq N$ . Write  $P = \{j_0, \dots, j_r\}$  in increasing order. If  $i \notin P$  then  $\varphi_{n+1-i}^w(q) = \sum_{t \in \{0, \dots, r\}: \frac{1}{j_t(j_t-1)}} \frac{1}{j_t(j_t-1)}$ . If  $i \in P$ , say  $i = j_s$ , then  $\varphi_{n+1-i}^w(q) = \frac{1}{j_s(j_s-1)} - \sum_{t \in \{0, \dots, r\}: \frac{1}{j_t(j_t-1)}} \frac{1}{j_t(j_t-1)}$ .*

Suppose that  $w$  is generated using a Balls and Bins process with probabilities  $p$ , where  $p$  is a SI sequence; then it stands to reason that if a sufficiently large number of balls is tossed (i.e.,  $m$  is large enough), then the voting power distribution under  $w$  will be very close to the power distribution under the weight vector  $p$ . This intuition is captured in the following lemma, which is proved in the appendix.

**LEMMA 5.4.** *Suppose that  $p = (p_1, \dots, p_n)$  is a SI sequence summing to 1, and let  $w_1, \dots, w_n$  be obtained by sampling  $m$  times from the distribution  $p_1, \dots, p_n$ .*

*Suppose that  $T \in (0, 1]$ , say  $T \in I^P(P)$  for some  $P \subseteq \{1, \dots, n\}$ . If the distance of  $T$  from the endpoints  $\tilde{p}(P), \tilde{p}(P^+)$  of  $I^P(P)$  is at least  $\Delta = \sqrt{\log(nm)}/m$  then with probability  $1 - \frac{2}{(nm)^2}$  it holds that if  $w$  is SI then for all  $i \in N$ ,  $\varphi_i^w(mT) = \varphi_i^P(T)$ .*

Combining both lemmas, we obtain our main result on the exponential case of the Balls and Bins distribution.

**THEOREM 5.5.** *Assume that  $m$  voters submit the votes according to the exponential distribution over candidates in which the probability the candidate  $i$  is voted for by each voter is proportional to  $\rho^{-i}$ , for some  $0 < \rho < 1/2$ . Assume further that  $m \geq C\rho^{-n}(2\rho - 1)^{-2} \log n$ , where  $C > 0$  is some global constant.*

*Suppose that  $T \in (0, 1]$ , say  $T \in I^P(P)$  for some  $P \subseteq \{1, \dots, n\}$ . If the distance of  $T$  from the endpoints  $\tilde{p}(P), \tilde{p}(P^+)$  of  $I^P(P)$  is at least  $\Delta = \sqrt{\log(nm)}/m$  then with probability  $1 - O(1/n)$  it holds that for all  $i \in \{1, \dots, n\}$ ,  $\varphi_i^w(mT) = \varphi_i^P(T)$ .*

*Furthermore, for all but finitely many values of  $T \in (0, 1]$ , the probability that  $\varphi_i^w(mT) = \varphi_i^P(T)$  tends to 1 as  $m \rightarrow \infty$ .*

**PROOF.** Lemma 5.2 gives a constant  $C > 0$  such that if  $m \geq C\rho^{-n}(2\rho - 1)^{-2} \log n$  then  $w$  is SI with probability  $1 - O(1/n)$ . Hence the first part of the theorem follows from Lemma 5.4.

For the second part, Lemma 5.2 shows that as  $m \rightarrow \infty$ , the probability that  $w$  is SI approaches 1. Suppose now that  $T$  is *not* of the form  $\tilde{p}(P)$  (these are the finitely many exceptions). When  $m$  is large enough, the conditions of Lemma 5.4 are satisfied, and so as  $m \rightarrow \infty$ , the error probability in that lemma goes to 0. The second part of the theorem follows.  $\square$

The theorem shows that in the case of the exponential distribution, if the number of balls is large enough then we can calculate with high probability the Shapley values of the resulting distribution based on the Shapley values of the original exponential distribution (without sampling). It therefore behooves us to study the Shapley values of an exponential distribution, or indeed any SI sequence.

## 6. SUPER-INCREASING SEQUENCES

In Section 5, we showed that studying the power distribution in the exponential balls and bins model boils down to the analysis of WVG's with super-increasing weights. In this section, we provide a comprehensive analysis of this case; in particular, we strongly generalize the results shown in [Zick et al. 2011] and [Zuckerman et al. 2012].

Up to this point, we assumed that the weights are arranged in non-decreasing order. In order to simplify our formulas, we will henceforth assume that the weights are ordered in non-increasing

order,  $w_1 > w_2 > \dots > w_n > 0$ . We also assume that  $w$  is a super-increasing sequence; that is, the sequence satisfying  $w_i > \sum_{j=i+1}^n w_j$  for all  $i \in N$ .

When considering different weight vectors, we will use  $\varphi_i^w(q)$  for the Shapley value of agent  $i$  under weight vector  $w$  and quota  $q$ .

### 6.1. Reducing Super-Increasing Weight Vectors to the Case of a Power Law of 2

Given a vector  $w$ , not every number  $q$  in the range  $(0, w(N)]$  can be written as a sum of members of  $\{w_1, \dots, w_n\}$ ; however, there are certain naturally defined intervals that partition  $(0, w(N)]$ . For a subset  $C \subseteq N$ , define  $\beta(C) = \sum_{i \in C} 2^{n-i}$ . Intuitively, we think of  $\beta(C)$  as the value resulting from the binary characteristic vector of the set of agents  $C$ . The purpose of the following two lemmas is to reduce every super-increasing weight vector to the case where the weights obey a power-law distribution, with a power of 2.

**LEMMA 6.1.** *Let  $w$  be a SI weight vector. For any  $S, T \subseteq N$ ,  $\beta(S) < \beta(T)$  if and only if  $w(S) < w(T)$ .*

**PROOF.** We first prove that if  $\beta(S) < \beta(T)$  then  $w(S) < w(T)$ . In order to prove this claim, it suffices to consider adjacent sets  $S, T \subseteq N$ , i.e., ones satisfying  $\beta(T) = \beta(S) + 1$ . Let  $\ell$  be the agent with the smallest weight that does not belong to  $S$ ; that is,  $\ell = \max\{i \in N \setminus S\}$ , and define  $C = S \cap \{1, \dots, \ell - 1\}$ ; that is,  $C$  is the set of all agents in  $S$  that have weight greater than  $w_\ell$ .

We claim that if  $\beta(T) = \beta(S) + 1$ , then  $S = C \cup \{\ell + 1, \dots, n\}$  and  $T = C \cup \{\ell\}$ . Indeed, suppose that  $S$  does not contain any element  $j \in \{\ell + 1, \dots, n\}$ ; then  $\ell$  is not the agent with the smallest weight that does not belong to  $S$ . The fact that  $T = C \cup \{\ell\}$  is an immediate consequence of the fact that  $\beta(T) = \beta(S) + 1$ .

Now,  $w(T) - w(S) = w_\ell - w(\{\ell + 1, \dots, n\})$ ; since  $w$  is super-increasing, it must be the case that  $w_\ell > \sum_{j=\ell+1}^n w_j$ , and in particular  $w(T) - w(S) > 0$ .

This shows that  $\beta(S) < \beta(T)$  implies  $w(S) < w(T)$ . Arrange now the  $2^n$  subsets of  $[N]$  according to  $\beta$ :  $\beta(S_0) < \dots < \beta(S_{2^n-1})$ . The preceding argument shows that also  $w(S_0) < \dots < w(S_{2^n-1})$ . It follows that the orders induced by  $\beta$  and by  $w$  are isomorphic, and so  $\beta(S) < \beta(T)$  if and only if  $w(S) < w(T)$ .  $\square$

For a non-empty set of agents  $C \subseteq N$ , we let  $C^- \subseteq N$  be the unique subset of agents satisfying  $\beta(C^-) = \beta(C) - 1$ . Lemma 6.1 shows that for every quota  $q \in (0, w(N)]$  there exists a *unique* set  $A(q) \subseteq N$  such that  $q$  is in  $(w(A(q)^-), w(A(q))]$ . Whenever we write  $A(q) = \{a_0, \dots, a_r\}$ , we will always assume that  $a_0 < \dots < a_r$ .

**LEMMA 6.2.** *Suppose that  $w$  is super-increasing; then for any  $i \in N$  and  $q \in (0, w(N)]$ ,  $\varphi_i^w(q) = \varphi_i^{\mathbf{b}}(\beta(A(q)))$ , where  $\mathbf{b} = (2^{n-1}, \dots, 1)$ .*

**PROOF.** Let  $\sigma$  be a random permutation in  $\text{Sym}_n$ , and recall that  $P_i(\sigma)$  is the set of agents appearing before agent  $i$  in  $\sigma$ . The Shapley value  $\varphi_i^w(q)$  is the probability that  $w(P_i(\sigma)) \in [q - w_i, q)$ , or equivalently, that  $q \in (w(P_i(\sigma)), w(P_i(\sigma)) + w_i]$ . Since the intervals  $(w(C^-), w(C)]$  partition  $(0, w(N)]$ ,  $q$  is in  $(w(P_i(\sigma)), w(P_i(\sigma)) + w_i]$  if and only if  $w(P_i(\sigma)) \leq w(A(q)^-)$  and  $w(A(q)) \leq w(P_i(\sigma) \cup \{i\})$ . Lemma 6.1 shows that this is equivalent to checking whether  $\beta(P_i(\sigma)) \leq \beta(A(q)^-)$  and  $\beta(A(q)) \leq \beta(P_i(\sigma) \cup \{i\})$ . Now, note that  $\beta(A(q)^-) = \beta(A(q)) - 1$ , so the above condition simply states that  $i$  is pivotal for  $\sigma$  under  $\mathbf{b}$  when the quota is  $\beta(A(q))$ .  $\square$

Lemma 6.2 implies that for any super-increasing  $w$ , if we wish to compute  $\varphi_i^w(q)$ , it is only necessary to find  $A(q)$ . However, finding  $A(q)$  is easy; a greedy algorithm can find  $A(q)$  in linear time (see Appendix C.1). In the special case in which  $w_i = d^{n-i}$  for some integer  $d$ , there is a particularly simple formula described in Appendix C.2.

We now present a closed-form formula for the Shapley values in the super-increasing case (the proof is given in the appendix). The resulting Shapley values are illustrated in Figure 3.

**THEOREM 6.3.** Consider an agent  $i \in N$  and a prescribed quota value  $q \in (0, w(N)]$ . Let  $A(q) = \{a_0, \dots, a_r\}$ . If  $i \notin A(q)$  then

$$\varphi_i(q) = \sum_{\substack{t \in \{0, \dots, r\} \\ a_t > i}} \frac{1}{a_t \binom{a_t-1}{t}}$$

If  $i \in A(q)$ , say  $i = a_s$ , then

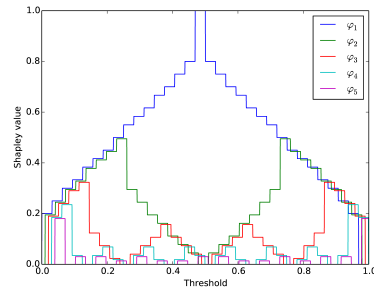
$$\varphi_i(q) = \frac{1}{a_s \binom{a_s-1}{s}} - \sum_{t > s} \frac{1}{a_t \binom{a_t-1}{t}}$$

**Example 6.4.** Consider a 10 agent game where  $w_i = 2^{n-i}$ . Let us compute the Shapley value of agent 7 when the quota is  $q = 27$ . We can write

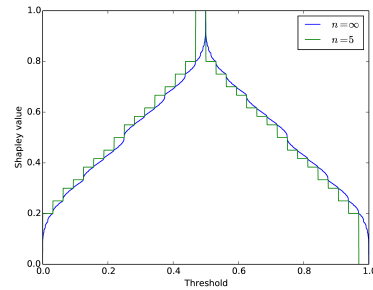
$$q = 16 + 8 + 2 + 1 = w_6 + w_7 + w_9 + w_{10},$$

hence  $A(q) = \{a_0 = 6, a_1 = 7, a_2 = 9, a_3 = 10\}$ . Since agent 7 is in  $A(q)$ , it must be the case that

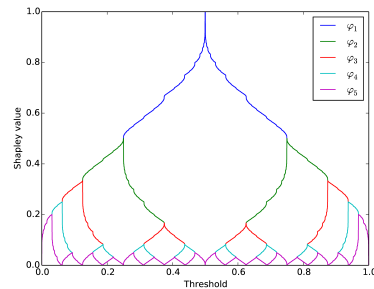
$$\varphi_7(27) = \frac{1}{7 \binom{6}{1}} - \frac{1}{9 \binom{8}{1}} - \frac{1}{10 \binom{9}{2}} \approx 0.007143.$$



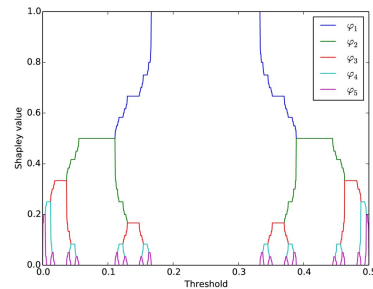
(a) Shapley values for  $n = 5$ ,  $w_i = 2^{-i}$ . Values  $\varphi_i(q)$  for different  $i$  are slightly nudged to show the effects of Lemma 6.6.



(b) Shapley values  $\varphi_1(q)$  for  $n = 5$ ,  $w_i = 2^{-i}$  compared to the limiting case  $n = \infty$ .



(c) Shapley values in the limiting case,  $w_i = 2^{-i}$ .



(d) Shapley values in the limiting case,  $w_i = 3^{-i}$ .

Fig. 3: Examples of several Shapley values corresponding to super-increasing sequences.

## 6.2. Properties of the Shapley values under Super-Increasing Weights

Zuckerman et al. [2012] prove a nice property of super-increasing sets:

**THEOREM 6.5** (LEMMA 19 IN ZUCKERMAN ET AL. [2012]). *Suppose that  $|N| \geq 3$ ; if the weights  $w$  are super-increasing then for every quota  $q \in (0, w(N)]$ , either  $\varphi_n(q) = \varphi_{n-1}(q)$  or  $\varphi_{n-1}(q) = \varphi_{n-2}(q)$ .*

In this section, we further generalize this result, using Theorem 6.3. Specifically, as a consequence of the theorem, we can determine in which cases  $\varphi_i(q) = \varphi_{i+1}(q)$ . The results are summarized in the following lemma, which is proved in the appendix.

**LEMMA 6.6.** *Given a quota  $q \in (0, w(N)]$ , let  $A(q) = \{a_0, \dots, a_r\}$ . Given some  $i \in N \setminus \{n\}$ , (a) if  $i, i+1 \in A(q)$  or  $i, i+1 \notin A(q)$  then  $\varphi_i(q) = \varphi_{i+1}(q)$ ; (b) if  $i \notin A(q)$  and  $i+1 \in A(q)$  then  $\varphi_i(q) \geq \varphi_{i+1}(q)$ , with equality if and only if  $i+1 = a_r$ ; (c) if  $i \in A(q)$  and  $i+1 \notin A(q)$  then  $\varphi_i(q) > \varphi_{i+1}(q)$ .*

Next, we show that Lemma 6.6 generalizes Theorem 6.5. We can, in fact, show the following stronger corollary.

**COROLLARY 6.7.** *Let  $w$  be a vector of super-increasing weights. Let  $A(q) = \{a_0, \dots, a_r\}$ . Then for all  $i \geq a_r$ , either  $\varphi_i(q) = \varphi_{i-1}(q)$ , or  $\varphi_{i-1}(q) = \varphi_{i-2}(q)$ .*

**PROOF.** Let  $\mathbb{I}(i \in S)$  be the indicator function for the property  $i \in S$ . Lemma 6.6 states that if  $\mathbb{I}(i \in A(q)) = \mathbb{I}(i-1 \in A(q))$  we have that  $\varphi_i(q) = \varphi_{i-1}(q)$ , and that if  $\mathbb{I}(i-1 \in A(q)) = \mathbb{I}(i-2 \in A(q))$  we have that  $\varphi_{i-1}(q) = \varphi_{i-2}(q)$ . Thus, if either holds, we are done.

Suppose that neither case holds. First, we consider the case that  $i-1 \notin A(q)$  but  $i \in A(q)$ . Since  $i \geq a_r$ , it must be the case that  $a_r = i$ . Invoking case (b) of Lemma 6.6 gives us that  $\varphi_{i-1}(q) = \varphi_i(q)$ .

Finally, suppose that  $i-1 \in A(q)$ . This means that  $i-2, i \notin A(q)$ . Since  $i-1 \in A(q)$  but  $i \notin A(q)$ , it must be the case that  $a_r = i-1$ . We can again invoke case (b) of Lemma 6.6 for  $i-2$  and  $i-1$ . Therefore, it must be the case that  $\varphi_{i-2}(q) = \varphi_{i-1}(q)$ , which concludes the proof.  $\square$

Invoking Corollary 6.7 with  $i = n$  gives Theorem 6.5.

We mention that it may be the case that  $\varphi_{i-1}(q) < \varphi_i(q) < \varphi_{i+1}(q)$  when weights are super-increasing.

**Example 6.8.** Let us observe the 10 agent game where for all  $i \in N = \{1, \dots, 10\}$ ,  $w_i = 2^{n-i}$ . As shown in Example 6.4,  $\varphi_7(27) \approx 0.007143$ . However,  $\varphi_8(27) \approx 0.005159$ , and  $\varphi_9(27) \approx 0.00119$ . The reason that  $\varphi_7(27) < \varphi_8(27) < \varphi_9(27)$  is the structure of  $A(27)$ . Recall that  $A(27) = \{6, 7, 9, 10\}$ ; that is,  $8 \notin A(27)$ , but 7 and 9 are in  $A(27)$ . However, there exists an element whose index is greater than 9 in  $A(27)$  (namely 10), so Corollary 6.7 does not hold.

Another interesting implication of Lemma 6.7 is the following. Suppose that  $A(q) = \{a_0, \dots, a_r\}$ , then for all  $i, j > a_r$ ,  $\varphi_i(q) = \varphi_j(q)$ .

A desirable property of parliaments that is discussed in [Zuckerman et al. 2012] is *separability*; if two parties have different weights, then they should have different voting power. More formally, if  $w_i > w_j$  then  $\varphi_i > \varphi_j$ . When weights are super-increasing, Theorem 6.5 shows that full separability cannot be achieved. Lemma 6.6 implies that agents are less separable under some quotas: if  $A(q)$  does not consist of low-weight agents, then low-weight agents are not separable under  $q$ . For example, in the case where  $w_i = 2^{n-i}$ , if  $q = \ell 2^{n-m}$ , where  $\ell$  is an odd number, then  $\varphi_n(q) = \varphi_{n-1}(q) = \dots = \varphi_{n-m+1}(q)$ .

Having gained a better understanding of when equality holds in the super-increasing case, let us proceed to bound the increase in power as the quota changes. Recall that given a set  $S \subseteq N$ ,  $S^-$  is the set for which  $\beta(S) = \beta(S^-) + 1$ . Since the Shapley values are constant in the interval  $(w(S^-), w(S)]$ , it follows that in order to analyze the behavior of  $\varphi_i(q)$ , one needs only determine

the rate of increase or decrease at quotas of the form  $w(S)$  for  $S \subseteq N$ . These are given by the following lemma, proved in the appendix.

LEMMA 6.9. *For any  $S \subseteq N$ , and any  $i \in N$ , if  $i \notin S^-$  then  $\varphi_i(w(S^-)) < \varphi_i(w(S))$ . If  $i \in S^-$  then  $\varphi_i(w(S^-)) > \varphi_i(w(S))$ .*

*Moreover,  $|\varphi_i(w(S)) - \varphi_i(w(S^-))| = \frac{1}{n}$  if one of the following holds: (a)  $S = \{n\}$ ; (b)  $i < n$  and  $S = \{1, \dots, i\}$  or  $S = \{i, n\}$ ; or (c)  $i = n$  and  $S = \{n-1\}$ . Otherwise,  $|\varphi_i(w(S)) - \varphi_i(w(S^-))| \leq \frac{1}{n(n-1)}$ .*

### 6.3. A Note on the Limiting Behavior of the Shapley Value under Super-Increasing Weights

Given a super-increasing sequence  $w_1, \dots, w_n$  (where again,  $w_1 > w_2 > \dots > w_n$ ) and some  $m \in N$ , let us write  $\mathbf{w}|_m$  for  $(w_1, \dots, w_m)$  and  $[m]$  for  $\{1, \dots, m\}$ . We write  $\varphi_i^{\mathbf{w}|_m}(q)$  for the Shapley value of agent  $i \in [m]$  in the weighted voting game in which the set of agents is  $[m]$ , the weights are  $\mathbf{w}|_m$ , and the quota is  $q$ . We also write  $A|_m(q)$  for the set  $S \subseteq [m]$  such that  $q \in (w|_m(S^-), w|_m(S))$ .

The following lemma relates  $\varphi_i^{\mathbf{w}}(q)$  and  $\varphi_i^{\mathbf{w}|_m}(q)$ .

LEMMA 6.10. *Let  $m \in N$  and  $i \in [m]$ , and let  $q \in (0, w([m]))$ . Then*

$$\varphi_i^{\mathbf{w}|_m}(q) = \varphi_i^{\mathbf{w}}(w(A|_m(q))).$$

PROOF. The proof makes use of Lemma 6.2. According to Lemma 6.2,  $\varphi_i^{\mathbf{w}}(q)$  is only a function of  $A(q)$ . Namely,  $\varphi_i^{\mathbf{w}}(q) = \varphi_i^{\mathbf{b}}(\beta(A(q)))$ , where  $\beta(S) = \sum_{i \in S} 2^{n-i}$ , and  $\mathbf{b}$  is the vector where  $b_i = 2^{n-i}$ . Now, on the one hand,  $\varphi_i^{\mathbf{w}|_m}(q) = \varphi_i^{\mathbf{b}}(\beta(A|_m(q)))$ . On the other hand, when  $q = w(A|_m(q))$ , then  $A(q)$  under the weight vector  $\mathbf{w}$  equals  $A|_m(q)$ . In particular,  $\varphi_i^{\mathbf{w}}(w(A|_m(q))) = \varphi_i^{\mathbf{b}}(\beta(A|_m(q)))$ , which concludes the proof.  $\square$

Therefore the plot of  $\varphi_i^{\mathbf{w}|_m}$  can be readily obtained from that of  $\varphi_i^{\mathbf{w}}$ . This suggests looking at the limiting case of an *infinite* super-increasing sequence  $(w_i)_{i=1}^{\infty}$ , which is a sequence satisfying  $w_i > 0$  and  $w_i \geq \sum_{j=i+1}^{\infty} w_j$  for all  $i \geq 1$ . In this section we make some normalizing assumptions that will be useful. Just like in the preceding subsections, we assume that weights are arranged in decreasing order; furthermore, we assume that  $w_1 = \frac{1}{2}$ . This is no loss of generality: it is an easy exercise to see that given a weight vector  $\mathbf{w}$  and some positive constant  $\alpha$ ,  $\varphi_i^{\mathbf{w}}(q) = \varphi_i^{\alpha\mathbf{w}}(\alpha q)$ . Thus, instead of the weight vector  $(2^{n-1}, 2^{n-2}, \dots, 1)$ , we now have  $(\frac{1}{2}, \frac{1}{4}, \dots, \frac{1}{2^{n-1}})$ .

The super-increasing condition implies that the infinite sequence sums to some value  $w(\infty) \leq 1$ . Lemma 6.10 suggests how to define  $\varphi_i(q)$  in this case: for  $q \in (0, w(\infty))$  and  $i \geq 1$ , let

$$\varphi_i^{(\infty)}(q) = \lim_{n \rightarrow \infty} \varphi_i^{\mathbf{w}|_n}(q).$$

We show that the limit exists by providing an explicit formula for it, as given in the main result of this section, Theorem 6.14. Under this definition, Lemma 6.10 easily extends to the case  $n = \infty$ :

LEMMA 6.11. *Let  $m \geq 1$  be an integer, let  $i \in [m]$ , and let  $q \in (0, w([m]))$ . Then  $\varphi_i^{\mathbf{w}|_m}(q) = \varphi_i^{(\infty)}(w(A|_m(q)))$ .*

PROOF. Lemma 6.10 shows that for  $n \geq m$ ,  $\varphi_i^{\mathbf{w}|_m}(q) = \varphi_i^{\mathbf{w}|_n}(w(A|_m(q)))$ , and therefore  $\varphi_i^{\mathbf{w}|_m}(q) = \lim_{n \rightarrow \infty} \varphi_i^{\mathbf{w}|_n}(w(A|_m(q))) = \varphi_i^{(\infty)}(w(A|_m(q)))$ .  $\square$

Below, we consider possibly infinite subsets  $S = \{a_0, \dots, a_r\}$  of the positive integers, ordered in increasing order; when  $r = \infty$ , the subset is infinite. Also, the notation  $\{a, \dots, \infty\}$  (or  $\{a, \dots, r\}$  when  $r = \infty$ ) means all integers larger than or equal to  $a$ .

Given a finite sequence of integers  $S = \{a_0, \dots, a_r\}$ , such that  $a_0 < a_1 < \dots < a_r$ , we define  $S^-$  to be  $\{a_0, \dots, a_{r-1}\} \cup \{a_{r+1}, \dots, \infty\}$ ; note the analogy to the finite case: when we had a

finite sequence of agents  $N$ ,  $S^-$  was the set such that  $w(S^-) < w(S)$ , but for all  $T \subseteq N$ , either  $w(T) \leq w(S^-)$  or  $w(T) \geq w(S)$ . This is also the case for  $S^-$  as defined above.

We start with some preliminary lemmas. For a (possibly infinite) subset  $S$  of the positive integers, define

$$\beta_\infty(S) = \sum_{i \in S} 2^{-i}.$$

First, we show an analog of Lemma 6.1.

**LEMMA 6.12.** *Suppose  $S, T \subseteq \mathbb{N}$  are two subsets of the positive integers. Then  $\beta_\infty(S) \leq \beta_\infty(T)$  if and only if  $w(S) \leq w(T)$ . Furthermore, if  $\beta_\infty(S) < \beta_\infty(T)$  then  $w(S) < w(T)$ .*

**PROOF.** Suppose that  $\beta_\infty(S) \leq \beta_\infty(T)$  and  $S \neq T$ . Let  $i = \min_{j \in T \setminus S} j$ ; then

$$w(T) - w(S) \geq w_i - \sum_{j=i+1}^{\infty} w_j \geq 0.$$

Equality is only possible if  $\max_{j \in T} j = i$  and  $S = T \setminus \{i\} \cup \{i+1, \dots, \infty\}$ . However, in that case  $\beta_\infty(S) = \beta_\infty(T)$ .  $\square$

There is a subtlety involved here: unlike the finite case explored in Lemma 6.1, we can have  $\beta_\infty(S) = \beta_\infty(T)$  for  $S \neq T$ . This is because dyadic rationals (numbers of the form  $\frac{a}{2^b}$  for some positive integer  $a$ ) have two different binary expansions. For example,  $\frac{1}{2} = (0.1000\dots)_2 = (0.0111\dots)_2$ . The lemma states (in this case) that  $w(\{1\}) \geq w(\{2, 3, 4, \dots\})$ , but there need not be equality.

Next, we will use the fact that any real  $r \in (0, 1)$  has a binary expansion with infinitely many 0s (alternatively, a set  $S_r$  such that  $\beta_\infty(S_r) = \sum_{n \in S_r} 2^{-n} = r$  and there are infinitely many  $n \notin S_r$ ), and a binary expansion with infinitely many 1s (alternatively, a set  $T_r$  such that  $\beta_\infty(T_r) = \sum_{n \in T_r} 2^{-n} = r$  and there are infinitely many  $n \in T_r$ ). If  $r$  is not dyadic, then it has a unique binary expansion which has infinitely many 0s and 1s. If  $r$  is dyadic, say  $r = \frac{1}{2}$ , then it has one expansion  $(0.1000\dots)_2$  with infinitely many 0s and another expansion  $(0.0111\dots)_2$  with infinitely many 1s.

The following lemma describes the analog of the intervals  $(w(S^-), w(S)]$  in the infinite case.

**LEMMA 6.13.** *Let  $q \in (0, w(\infty))$ . There exists a non-empty subset  $S$  of the positive integers such that either  $q = w(S)$  or  $S = \{a_0, \dots, a_r\}$  is finite and  $q \in (w(S^-), w(S)]$ .*

**PROOF.** Since  $q < w(\infty)$ , there exists some finite  $m$  such that  $q \leq w([m])$ . For any  $n \geq m$ , let  $A|_n = A|_n(q)$ . Let  $Q|_n$  be the subset of  $[n]$  preceding  $A|_n$ , and let  $R|_n$  be the subset of  $[n+1]$  preceding  $A|_n$ ; here “preceding” is in the sense of  $X \mapsto X^-$ . The interval  $(w(Q|_n), w(A|_n)]$  splits into  $(w(Q|_n), w(R|_n)] \cup (w(R|_n), w(A|_n)]$ , and so  $A|_{n+1} \in \{R|_n, A|_n\}$ . Also  $\beta_\infty(A|_{n+1}) \leq \beta_\infty(A|_n)$ , with equality only if  $A|_{n+1} = A|_n$ .

We consider two cases. The first case is when for some integer  $M$ , for all  $n \geq M$  we have  $A|_n = A = \{a_0, \dots, a_r\}$ . In that case for all  $n \geq M$ ,

$$\sum_{t=0}^{r-1} w_{a_t} + \sum_{t=a_r+1}^n w_t < q \leq \sum_{t=0}^r w_{a_t},$$

and taking the limit  $n \rightarrow \infty$  we obtain  $q \in (w(A^-), w(A)]$ .

The other case is when  $A|_n$  never stabilizes. The sequence  $\beta_\infty(A|_n)$  is monotonically decreasing, and reaches a limit  $b$  satisfying  $b < \beta_\infty(A|_n)$  for all  $n$ . Since  $w(A|_m) \in (w(Q|_n), w(A|_n)]$  for all integers  $m \geq n \geq 1$ , Lemma 6.12 implies that  $b \in [\beta_\infty(Q|_n), \beta_\infty(A|_n))$ .

Let  $L$  be a subset such that  $b = \beta_\infty(L)$  and there are infinitely many  $i \notin L$ , and define  $L|_n = L \cap [n]$ . We have  $b \in [\beta_\infty(L|_n), \beta_\infty(L|_n) + 2^{-n}]$ . Therefore  $Q|_n = L|_n$ , and so  $q > w(Q|_n) = w(L|_n)$ . Taking the limit  $n \rightarrow \infty$ , we deduce that  $q \geq w(L)$ .

If  $n \notin L$  then  $A|_n = Q|_n \cup \{n\}$ , and so  $q \leq w(A|_n) = w(L|_n) + w_n$ . Since there are infinitely many such  $n$ , taking the limit  $n \rightarrow \infty$  we conclude that  $q \leq w(L)$  and so  $q = w(L)$ .  $\square$

We can now give an explicit formula for  $\varphi_i^{(\infty)}$ . We extend the notation used in the preceding subsections to accommodate the notions given in Lemma 6.13. Given some  $q \in (0, w(\infty))$ , we write  $A(q) \subseteq \mathbb{N}$  to be an infinite set  $S$  such that  $q = w(S)$ , or the finite set  $S$  for which  $q \in (w(S^-), w(S)]$ . Observe that in the first case there may be more than one set  $S$  such that  $q = w(S)$ ; Theorem 6.14 holds for any of the possible representations of  $q$  using  $w$ .

**THEOREM 6.14.** *Let  $q \in (0, w(\infty))$  and let  $i$  be a positive integer. Let  $A(q) = \{a_0, \dots, a_r\}$  be the set defined in Lemma 6.13. Then*

- (a) *the limit  $\varphi_i^{(\infty)}(q) = \lim_{n \rightarrow \infty} \varphi_i^{w|_n}(q)$  exists.*  
 (b) *if  $i \notin A(q)$  then*

$$\varphi_i^{(\infty)}(q) = \sum_{\substack{t \in \{0, \dots, r\}: \\ a_t > i}} \frac{1}{a_t \binom{a_t-1}{t}},$$

and if  $i \in A(q)$  — say  $i = a_s$  — then

$$\varphi_i^{(\infty)}(q) = \frac{1}{a_s \binom{a_s-1}{s}} - \sum_{\substack{t \in \{0, \dots, r\}: \\ a_t > i}} \frac{1}{a_t \binom{a_t-1}{t-1}}.$$

**PROOF.** We comment that the convergence of the sums in the theorem is guaranteed by Lemma C.4. We again write  $A(q) = S = \{a_0, \dots, a_r\}$ .

Suppose first that  $S$  is finite; according to Lemma 6.12,  $q \in (w(S^-), w(S)]$ . Let  $j^*$  be  $\max_{j \in S} j$ ; then for all  $n > j^*$ ,  $A|_n(q) = S$ , and so Lemma 6.10 shows that  $\varphi_i^{w|_n}(q) = \varphi_i^{w|_{j^*}}(q)$ . Therefore the limit exists and equals the stated formula, which is the same as the one given by Theorem 6.3. Thus, we have covered the case where  $S$  is finite.

Suppose next that  $S$  is infinite; by Lemma 6.12,  $q = w(S)$ . Consider first the case in which we can also write  $q = w(Q)$  for some finite  $Q$ , say  $Q = \{q_0, \dots, q_u\}$  (think again of the case of  $q = \frac{1}{2}$ , which can be represented by either  $\{1\}$  or  $\{2, 3, 4, \dots\}$ ). Then  $S = \{q_0, \dots, q_{u-1}\} \cup \{q_u + 1, q_u + 2, \dots, \infty\}$ . We now consider several cases.

If  $i < q_u$  and  $i \notin S$ , then  $i \notin Q$  and

$$\varphi_i^{(\infty)}(q) = \sum_{\substack{t \in \{0, \dots, u\}: \\ q_t > i}} \frac{1}{q_t \binom{q_t-1}{t}} = \sum_{\substack{t \in \{0, \dots, u-1\}: \\ q_t > i}} \frac{1}{q_t \binom{q_t-1}{t}} + \sum_{\ell=1}^{\infty} \frac{1}{(q_u + \ell) \binom{q_u + \ell - 1}{t + \ell - 1}},$$

using Lemma C.4. The right-hand side is the expression we gave for  $\varphi_i^{(\infty)}(w(S))$ .

If  $i < q_u$  and  $i \in S$ , say  $i = q_s$ , then  $i \in Q$  and

$$\varphi_i^{(\infty)}(q) = \frac{1}{i \binom{i-1}{s}} - \sum_{\substack{t \in \{0, \dots, u\}: \\ q_t > i}} \frac{1}{q_t \binom{q_t-1}{t}} = \frac{1}{i \binom{i-1}{s}} - \sum_{\substack{t \in \{0, \dots, u-1\}: \\ q_t > i}} \frac{1}{q_t \binom{q_t-1}{t}} - \sum_{\ell=1}^{\infty} \frac{1}{(q_u + \ell) \binom{q_u + \ell - 1}{t + \ell - 1}},$$

using Lemma C.4. The right-hand side is the expression we gave for  $\varphi_i^{(\infty)}(w(S))$ .

If  $i = q_u$  then  $i \in Q$  and  $i \notin S$ . In that case

$$\varphi_i^{(\infty)}(q) = \frac{1}{i \binom{i-1}{u}} = \sum_{\ell=1}^{\infty} \frac{1}{(i + \ell) \binom{i + \ell - 1}{u + \ell - 1}},$$

using Lemma C.4. The right-hand side is the expression we gave for  $\varphi_i^{(\infty)}(w(S))$ .



Finally, if  $i > q_u$  then  $i \notin Q$  and  $i \in S$ . Suppose that  $i$  is the  $v$ -th member in  $S$ . In that case

$$\varphi_i^{(\infty)}(q) = 0 = \frac{1}{i \binom{i-1}{v}} - \sum_{\ell=1}^{\infty} \frac{1}{(i+\ell) \binom{i+\ell-1}{v+\ell-1}},$$

using Lemma C.4. The right-hand side is the expression we gave for  $\varphi_i^{(\infty)}(w(S))$ .

It remains to consider the case in which  $q$  cannot be written as  $q = w(Q)$  for finite  $Q$ . In that case, there are infinitely many positive integers  $n$  such that  $n \in S$  and infinitely many such that  $n \notin S$ . This implies that for every positive integer  $n$ ,  $q \in (w(S \cap [n]), w(S \cap [n]) + w_n)$ , and so  $S|_n^-(q) = S \cap [n]$ . Lemma 6.9 shows that  $|\varphi_n(q) - \varphi_n(w(S \cap [n]))| \leq \frac{1}{n}$ . On the other hand, Theorem 6.3 readily implies that  $\varphi_n(w(S \cap [n]))$  tends to the expression we gave for  $\varphi_i^{(\infty)}(w(S))$ . We conclude that  $\varphi_n(q)$  tends to the same expression.  $\square$

We conclude by showing that the limiting functions  $\varphi_i^{(\infty)}$  are continuous (see appendix for the proof).

**THEOREM 6.15.** *Let  $i$  be a positive integer. The function  $\varphi_i^{(\infty)}$  is continuous on  $(0, w(\infty))$ , and  $\lim_{q \rightarrow 0} \varphi_i^{(\infty)}(q) = \lim_{q \rightarrow w(\infty)} \varphi_i^{(\infty)}(q) = 0$ .*

Summarizing, we can extend the functions  $\varphi_i^{w|n}$  to a continuous function  $\varphi_i^{(\infty)}$  which agrees with  $\varphi_i^{w|n}$  on the points  $w(S)$  for  $S \subseteq \{1, \dots, n\}$ .

When  $w_i = 2^{-i}$  the plot of  $\varphi^{(\infty)}$  has no flat areas, but when  $w_i = d^{-i}$  for  $d > 2$ , the limiting function is constant on intervals  $(w(S^-), w(S))$ . This is reflected in Figure 3. These flat areas highlight a curious phenomenon. Some people find it hard to accept that  $0.999\dots = 1$ . Our setting demonstrates a case in which this equality indeed fails. When  $w_1 > \sum_{j=2}^{\infty} w_j$ , we have  $w(\{2, 3, \dots, \infty\}) < w(\{1\})$ , which corresponds to the strict inequality  $0.0111\dots < 0.1$  in binary, or  $0.4999\dots < 0.5$  in decimal. The infinitesimal difference is expanded to an interval  $(w(\{1\}^-), w(\{1\}))$  of non-zero width  $w_1 - \sum_{j=2}^{\infty} w_j$ . When  $w_i > \sum_{j=i+1}^{\infty} w_j$  for all  $i$ , this phenomenon happens around every dyadic number.

## 7. CONCLUSIONS AND FUTURE WORK

We have studied the Shapley value as a function of the quota under a number of natural weight distributions. Assuming that weights are drawn from balls and bins distributions allows us to reason rigorously about the effect of quota changes. We were also able to completely characterize the case where weights are super-increasing, strongly generalizing previous work. The take-home message from our work is that changes to the quota matter, even when weights are nearly identical. Given the relative success of this analysis, it would be interesting to study other natural weight distributions (the case of i.i.d. weights is studied by Filmus, Oren, and Soundararajan [Filmus et al.]). Moreover, our results show that employing probabilistic approaches to cooperative games (beyond the case of WVG's) may be a useful research avenue.

## REFERENCES

- BACHRACH, Y., MARKAKIS, E., RESNICK, E., PROCACCIA, A., ROSENSCHEIN, J., AND SABERI, A. 2010. Approximating power indices: theoretical and empirical analysis. *Autonomous Agents and Multi-Agent Systems* 20, 2, 105–122.
- BANZHAF, J. 1964. Weighted voting doesn't work: a mathematical analysis. *Rutgers Law Review* 19, 317.
- CHALKIADAKIS, G., ELKIND, E., AND WOOLDRIDGE, M. 2011. *Computational Aspects of Cooperative Game Theory*. Morgan and Claypool.

- FATIMA, S., WOOLDRIDGE, M., AND JENNINGS, N. 2008. A linear approximation method for the shapley value. *Artificial Intelligence* 172, 14, 1673–1699.
- FELSENTHAL, D. AND MACHOVER, M. 2005. Voting power measurement: a story of misreinvention. *Social choice and welfare* 25, 2, 485–506.
- FELSENTHAL, D. S. AND MACHOVER, M. 1998. *The Measurement of Voting Power: Theory and Practice, Problems and Paradoxes*. Edward Elgar Publishing.
- FILMUS, Y., OREN, J., AND SOUNDARARAJAN, K. Shapley values in weighted voting games with random weights. *Submitted*.
- LEECH, D. 2002. Designing the voting system for the council of the european union. *Public Choice* 113, 437–464. 10.1023/A:1020877015060.
- LEECH, D. AND MACHOVER, M. 2003. Qualified majority voting: the effect of the quota. *LSE Research Online*.
- LINDNER, I. 2004. Power measures in large weighted voting games. Ph.D. thesis, Univ. of Hamburg.
- MANN, I. AND SHAPLEY, L. 1960. Values of large games IV: Evaluating the electoral college by monte-carlo techniques. Tech. rep., The RAND Corporation.
- MANN, I. AND SHAPLEY, L. 1962. Values of large games VI: Evaluating the electoral college exactly. Tech. rep., The RAND Corporation.
- MASCHLER, M., SOLAN, E., AND ZAMIR, S. 2013. *Game Theory*. Cambridge University Press.
- MITZENMACHER, M., RICHA, A. W., AND SITARAMAN, R. 2000. The power of two random choices: A survey of techniques and results. In *Handbook of Randomized Computing*. Kluwer, 255–312.
- MITZENMACHER, M. AND UPFAL, E. 2005. *Probability and computing - randomized algorithms and probabilistic analysis*. Cambridge University Press.
- PELEG, B. AND SUDHÖLTER, P. 2007. *Introduction to the Theory of Cooperative Games* Second Ed. Theory and Decision Library. Series C: Game Theory, Mathematical Programming and Operations Research Series, vol. 34. Springer, Berlin.
- PENROSE, L. 1946. The elementary statistics of majority voting. *Journal of the Royal Statistical Society* 109, 1, 53–57.
- RAAB, M. AND STEGER, A. 1998. Balls into bins: A simple and tight analysis. In *Randomization and Approximation Techniques in Computer Science*, M. Luby, J. Rolim, and M. Serna, Eds. Lecture Notes in Computer Science Series, vol. 1518. Springer Berlin Heidelberg, 159–170.
- SHAPLEY, L. 1953. A value for  $n$ -person games. In *Contributions to the Theory of Games, vol. 2*. Annals of Mathematics Studies, no. 28. Princeton University Press, Princeton, N. J., 307–317.
- SHAPLEY, L. AND SHUBIK, M. 1954. A method for evaluating the distribution of power in a committee system. *The American Political Science Review* 48, 3, 787–792.
- SŁOMCZYŃSKI, W. AND ŻYCZKOWSKI, K. 2006. Penrose voting system and optimal quota. *Acta Physica Polonica B* 37, 11, 3133–3143.
- TAUMAN, Y. AND JELNOV, A. 2012. Voting power and proportional representation of voters. Dept. of Econs. Working Papers 12–04, Stony Brook Univ.
- ZICK, Y. 2013. On random quotas and proportional representation in weighted voting games. In *Proceedings of the 23rd International Joint Conference on Artificial Intelligence (IJCAI-13)*. 432–438.
- ZICK, Y., SKOPALIK, A., AND ELKIND, E. 2011. The shapley value as a function of the quota in weighted voting games. In *Proceedings of the Twenty-Second International Joint Conference on Artificial Intelligence (IJCAI-11)*. 490–495.
- ZUCKERMAN, M., FALISZEWSKI, P., BACHRACH, Y., AND ELKIND, E. 2012. Manipulating the quota in weighted voting games. *Artificial Intelligence* 180–181, 1–19.

## Appendix

**A. MISSING PROOFS FROM SECTION 4****A.1. Proof of Lemma 4.1**

We now provide the complete proof of Lemma 4.1.

LEMMA 4.1. *For all agents  $i, j \in N$ ,*

$$|\varphi_j - \varphi_i| = \frac{1}{n-1} \sum_{\ell=0}^{n-2} \Pr_{S \in \mathcal{R}^{[N \setminus \{i, j\}]}} [q - \max(w_i, w_j) \leq w(S) < q - \min(w_i, w_j)].$$

PROOF. Assume without loss of generality that  $w_j \geq w_i$ , and so  $\varphi_j \geq \varphi_i$ . For  $\sigma \in \text{Sym}_n$ , let  $T_{ij}(\sigma)$  be the permutation obtained by exchanging agents  $i$  and  $j$ . Then by the definition of the Shapley value and by linearity of expectations:

$$\begin{aligned} \varphi_j - \varphi_i &= \mathbb{E}_{\sigma \in \text{Sym}_n} (m_j(\sigma) - m_i(\sigma)) \\ &= \mathbb{E}_{\sigma \in \text{Sym}_n} m_j(\sigma) - \mathbb{E}_{\sigma \in \text{Sym}_n} m_i(\sigma) = \mathbb{E}_{\sigma \in \text{Sym}_n} (m_j(T_{ij}(\sigma)) - m_i(\sigma)). \end{aligned}$$

We proceed to evaluate  $m_j(T_{ij}(\sigma)) - m_i(\sigma)$ . Suppose first that agent  $i$  precedes agent  $j$  in  $\sigma$ , so that  $\sigma = S i R j U$  and  $T_{ij}(\sigma) = S j R i U$  (where  $S, R$ , and  $U$  form a partition of  $N \setminus \{i, j\}$ ). In this case  $m_j(T_{ij}(\sigma)) - m_i(\sigma) \neq 0$  precisely when  $w(S) + w_i < q \leq w(S) + w_j$ , in which case  $m_j(T_{ij}(\sigma)) - m_i(\sigma) = 1$ ; we can rewrite the condition as  $w(S) \in [q - w_j, q - w_i)$ .

When agent  $j$  precedes agent  $i$  in  $\sigma$ , we can write  $\sigma = S j R i U$  and  $T_{ij}(\sigma) = S i R j U$ . In this case  $m_j(T_{ij}(\sigma)) - m_i(\sigma) \neq 0$  precisely when  $w(S) + w_i + w(R) < q \leq w(S) + w_j + w(R)$ , in which case  $m_j(T_{ij}(\sigma)) - m_i(\sigma) = 1$ ; we can rewrite the condition as  $w(S \cup R) \in [q - w_j, q - w_i)$ .

In order to unify both conditions together, define  $P'_i(\sigma) = P_i(\sigma) \setminus \{j\}$ . Using this definition, we see that  $m_j(T_{ij}(\sigma)) - m_i(\sigma)$  is the indicator of the event  $w(P'_i(\sigma)) \in [q - w_j, q - w_i)$ . The cardinality  $|P'_i(\sigma)|$  is exactly the position of agent  $i$  in the permutation  $\sigma'$  obtained by removing agent  $j$  from  $\sigma$ , minus one. Since  $\sigma$  is a uniformly random permutation of  $N$ ,  $\sigma'$  is a uniformly random permutation of  $N \setminus \{j\}$ , and so  $|P'_i(\sigma)|$  is distributed randomly among  $\{0, \dots, n-2\}$ . Given  $|P'_i(\sigma)|$ , the set  $P_i(\sigma)$  is chosen randomly among all subsets of  $N \setminus \{i, j\}$  of the specified size, yielding our formula.  $\square$

**A.2. Proof of Theorem 4.2**

We give the full proof of the following theorem.

THEOREM 4.2. *Let  $M = \frac{m}{3n^3}$ . Suppose that  $|q - \frac{\ell m}{n}| > \frac{1}{\sqrt{M}} \frac{m}{n}$  for all integers  $\ell$ . Then with probability  $1 - 2(\frac{2}{e})^n$ , all Shapley values are equal to  $1/n$ .*

In this section, we do not assume that the weights  $w_1, \dots, w_n$  are ordered, in order to maintain the fact that the weights are independent random variables.

The idea of the proof is to use the following criterion, which is a consequence of Lemma 4.1:

PROPOSITION A.1. *Suppose that for all agents  $i, j \in N$  and for all subsets  $S \subseteq N \setminus \{i, j\}$ , we have  $q \notin (w(S \cup \{i\}), w(S \cup \{j\}))$ . Then all Shapley values are equal to  $1/n$ .*

PROOF. We show that under the assumption on  $q$ , all Shapley values are equal, and so all must equal  $1/n$ . Suppose that for some agents  $i \neq j$ , we have  $\varphi_i < \varphi_j$  (and so  $w_i < w_j$ ). Lemma 4.1 implies the existence of a set  $S \subseteq N \setminus \{i, j\}$  satisfying  $q - w_j \leq w(S) < q - w_i$ , or in other words  $w(S) + w_i < q \leq w(S) + w_j$ . This is exactly what is ruled out by the assumption on  $q$ .  $\square$

Next, we show that the weights  $w(S)$  are concentrated around points of the form  $\ell \frac{m}{n}$ .

LEMMA A.2. *Suppose that  $m > 3n^2$ . With probability  $1 - 2(\frac{2}{e})^n$ , the following holds: for all  $S \subseteq N$ ,  $|w(S) - \frac{|S|m}{n}| \leq \sqrt{3nm}$ .*

PROOF. The proof uses a straightforward Chernoff bound. We can assume that  $S \neq \emptyset$  (as otherwise the bound is trivial). For each non-empty set  $S \subseteq N$ , the distribution of  $w(S)$  is  $B(m, \frac{|S|}{n})$ . Therefore for  $0 < \delta < 1$ ,

$$\Pr \left[ \left| w(S) - \frac{|S|m}{n} \right| > \delta \frac{|S|m}{n} \right] \leq 2e^{-\frac{\delta^2 |S|m}{3n}}.$$

Choosing  $\delta = \sqrt{\frac{3n^2}{|S|m}} < 1$ , we obtain

$$\Pr \left[ \left| w(S) - \frac{|S|m}{n} \right| > \sqrt{3|S|m} \right] \leq 2e^{-n}.$$

Since there are  $2^n$  possible sets  $S$ , a union bound implies that  $|w(S) - \frac{|S|m}{n}| \leq \sqrt{3nm}$  with probability at least  $1 - 2(\frac{2}{e})^n$ .  $\square$

Finally, we require the following simple property of quotas.

PROPOSITION A.3. *Let  $n \leq m$  be two integers, then for any  $q \in (0, m]$ , there exists some  $\ell \leq n$  such that  $|q - \ell m/n| \geq m/n$ .*

PROOF. For any  $q \in (0, m]$ , there exists some  $\ell^*$  such that  $q \in (\ell^* m/n, (\ell^* + 1)m/n]$ ; in particular, it is either the case that  $|q - \ell^* m/n| < m/n$  or  $|q - (\ell^* + 1)m/n| < m/n$ .  $\square$

Theorem 4.2 is an immediate corollary of the above claims, as we now show.

PROOF OF THEOREM 4.2. First, note that  $M > 1$ , as otherwise, it would imply that for all  $\ell = 1, \dots, n$ ,  $|q - \ell m/n| > \frac{1}{\sqrt{M}} \frac{m}{n} \geq \frac{m}{n}$ . This is impossible according to Proposition A.3. Thus,  $M > 1$ , i.e.,  $m > 3n^3 \geq 3n^2$ .

Lemma A.2 shows that when  $m > 3n^3$ , with probability  $1 - 2(\frac{2}{e})^n$ , for all sets  $S$  we have  $|w(S) - \frac{|S|m}{n}| \leq \sqrt{3nm}$ . Condition on this event. Suppose, for the sake of obtaining a contradiction, that  $\varphi_i < \varphi_j$  for some agents  $i, j$ . Then Proposition A.1 shows that there must exist some  $S \subseteq N \setminus \{i, j\}$  such that  $q \in (w(S \cup \{i\}), w(S \cup \{j\})]$ . Since both  $w(S \cup \{i\})$  and  $w(S \cup \{j\})$  are  $\sqrt{3nm}$ -close to  $\frac{(|S|+1)m}{n}$ , this implies that  $|q - \frac{(|S|+1)m}{n}| \leq \sqrt{3nm}$ . However,

$$\sqrt{3nm} = \frac{m}{n} \sqrt{\frac{3n^3 m}{m^2}} = \sqrt{\frac{3n^3}{m}} \cdot \frac{m}{n} = \frac{1}{\sqrt{M}} \cdot \frac{m}{n},$$

contradicting our assumption that  $|q - \ell \frac{m}{n}| \geq \frac{1}{\sqrt{M}} \cdot \frac{m}{n}$  for all  $\ell$ . We conclude that with probability at least  $1 - 2(\frac{2}{e})^n$ , all agents have the same Shapley value  $1/n$ .  $\square$

### A.3. Proof of Lemma 4.5

We prove the following lemma.

LEMMA 4.5. *Let  $q = \ell \cdot \frac{m}{n}$ , where  $\ell \in \{1, \dots, n-1\}$ . For  $m = \Omega(n^3 \log n)$ ,*

$$\mathbb{E}[\varphi_1] = \frac{1}{2(n-\ell)} - \frac{\ell}{n(n-\ell)} + \frac{1}{n-\ell} \Pr_{A \in \mathcal{R}_{\left[ \begin{smallmatrix} N \\ \ell-1 \end{smallmatrix} \right]}} [w(A) + w_1 \geq q] \pm O\left(\frac{1}{n^2}\right).$$

We will need the fact that with high probability,  $w_1$  is close to  $m/n$ .

LEMMA A.4. *With probability at least  $1 - 1/n$ ,*

$$w_1 \in \left[ \frac{m}{n} - \sqrt{\frac{4m \log n}{n}}, \frac{m}{n} \right].$$

PROOF. Clearly  $w_1 \leq m/n$  always, so we only need to address the lower bound on  $w_1$ . Let  $w'_1, \dots, w'_n$  be the loads of the bins before sorting them. The loads  $w'_i$  are independent random variables with distribution  $B(m, 1/n)$ . For each index  $i$ , Chernoff's bound shows that

$$\Pr \left[ w'_i < \frac{m}{n} - \sqrt{\frac{4m \log n}{n}} \right] \leq e^{-\frac{(4m \log n)/n}{2m/n}} = e^{-2 \log n} = \frac{1}{n^2}.$$

A union bound shows that with probability  $1 - 1/n$ , all  $i \in N$  satisfy  $w'_i \geq \frac{m}{n} - \sqrt{\frac{4m \log n}{n}}$ , and in particular  $w_1 \geq \frac{m}{n} - \sqrt{\frac{4m \log n}{n}}$ .  $\square$

Below we will be interested in bounding probabilities of the form  $\Pr_{A \in R \binom{N \setminus \{1\}}{k}} [P(w(A))]$  for predicates  $P$ . The following lemma shows how to bound these probabilities from above.

LEMMA A.5. *For a weight vector  $\mathbf{w}$  and  $S \subseteq N$ , let  $\mathcal{E}(w(S))$  be a random event (i.e., some predicate on  $w(S)$ ), and let  $0 \leq k \leq n - 1$ . Then*

$$\Pr_{A \in R \binom{N \setminus \{1\}}{k}} [\mathcal{E}(w(A))] \leq \frac{n}{n-k} \Pr[\mathcal{E}(B(m, \frac{k}{n}))].$$

Also,

$$\Pr_{A \in R \binom{N}{k}} [\mathcal{E}(w(A))] = \Pr[\mathcal{E}(B(m, \frac{k}{n}))].$$

PROOF. First, we have

$$\begin{aligned} \Pr_{A \in R \binom{N \setminus \{1\}}{k}} [\mathcal{E}(w(A))] &= \frac{1}{\binom{n-1}{k}} \sum_{A \in \binom{N \setminus \{1\}}{k}} \Pr[\mathcal{E}(w(A))] \\ &\leq \frac{1}{\binom{n-1}{k}} \sum_{A \in \binom{N}{k}} \Pr[\mathcal{E}(w(A))] \\ &= \frac{n}{n-k} \Pr_{A \in R \binom{N}{k}} [\mathcal{E}(w(A))]. \end{aligned}$$

Consider the last expression. Since the probability is over all subsets of  $N$  of size  $k$ , the same value is obtained from the *unsorted* Balls and Bins process (without sorting the loads). Under this process,  $w(A) \sim B(m, \frac{k}{n})$  for all  $A \in \binom{N}{k}$ , and so

$$\Pr_{A \in R \binom{N}{k}} [\mathcal{E}(w(A))] = \Pr_{w \sim B(m, \frac{k}{n})} [\mathcal{E}(w)].$$

This implies the lemma.  $\square$

Let  $p_k = \Pr_{A \in R \binom{N \setminus \{1\}}{k}} [q - w_1 \leq w(A) < q]$ , and recall that formula (1) shows that  $\varphi_1 = \frac{1}{n} \sum_{k=0}^{n-1} p_k$ . We start by showing that the only non-negligible  $p_k$  are  $p_{\ell-1}$  and  $p_\ell$ , using a Chernoff bound. The idea is that when  $k \geq \ell + 1$ , it is highly unlikely that  $w(A) < q$ , and when  $k \leq \ell - 1$ , it is highly unlikely that  $w(A) \geq q - w_1$ .

LEMMA A.6. *Suppose that  $m \geq 9n^2 \log n$ . Then for  $k \in \{1, \dots, n\} \setminus \{\ell - 1, \ell\}$  we have  $p_k \leq 1/n^2$ , and so*

$$0 \leq \mathbb{E}[\varphi_1] - \frac{p_{\ell-1} + p_\ell}{n} \leq \frac{1}{n^2}.$$

PROOF. Let  $k \in N$ . Lemma A.5 shows that

$$p_k \leq n \Pr[q - w_1 \leq B(m, \frac{k}{n}) < q].$$

Suppose first that  $k \geq \ell + 1$ . Chernoff's bound shows that

$$\Pr[q - w_1 \leq B(m, \frac{k}{n}) < q] \leq \Pr[B(m, \frac{k}{n}) < \frac{km}{n} - \frac{m}{n}] \leq e^{-\frac{(m/n)^2}{3km/n}} = e^{-m/(3nk)} \leq \frac{1}{n^3}.$$

Suppose next that  $k \leq \ell - 2$ . Since  $w_1 \leq m/n$ , another application of Chernoff's bound gives

$$\begin{aligned} \Pr[q - w_1 \leq B(m, \frac{k}{n}) < q] &\leq \Pr[B(m, \frac{k}{n}) \geq \frac{(\ell-1)m}{n}] \\ &\leq \Pr[B(m, \frac{k}{n}) \geq \frac{km}{n} + \frac{m}{n}] \\ &\leq e^{-\frac{(m/n)^2}{3km/n}} = e^{-m/(3nk)} \leq \frac{1}{n^3}. \end{aligned}$$

Therefore  $p_k \leq 1/n^2$  for all  $k \in N \setminus \{\ell - 1, \ell\}$ . The estimate for  $\mathbb{E}[\varphi_1]$  follows from formula (1).  $\square$

The next step is to consider the following estimates for  $p_{\ell-1}, p_\ell$ :

$$\begin{aligned} p'_{\ell-1} &= \Pr_{A \in_R \left[ \binom{N \setminus \{\ell-1\}}{\ell-1} \right]} [q - w_1 \leq w(A)], \\ p'_\ell &= \Pr_{A \in_R \left[ \binom{N \setminus \{\ell\}}{\ell} \right]} [w(A) < q]. \end{aligned}$$

The following lemma shows that  $p'_{\ell-1} \approx p_{\ell-1}$  and  $p'_\ell \approx p_\ell$ .

LEMMA A.7. *Suppose that  $m \geq 24n^2 \log n$ . Then  $p_{\ell-1} \leq p'_{\ell-1} \leq p_{\ell-1} + \frac{1}{n}$  and  $p_\ell \leq p'_\ell \leq p_\ell + \frac{2}{n}$ , and so*

$$-\frac{3}{n^2} \leq \mathbb{E}[\varphi_1] - \frac{p'_{\ell-1} + p'_\ell}{n} \leq \frac{1}{n^2}.$$

PROOF. Clearly  $p_{\ell-1} \leq p'_{\ell-1}$  and  $p_\ell \leq p'_\ell$ . First,

$$p'_{\ell-1} - p_{\ell-1} \leq \Pr_{A \in_R \left[ \binom{N \setminus \{\ell-1\}}{\ell-1} \right]} [w(A) \geq q] \leq n \Pr[B(m, \frac{\ell-1}{n}) \geq q],$$

using Lemma A.5. Chernoff's bound shows that

$$\Pr[B(m, \frac{\ell-1}{n}) \geq \frac{(\ell-1)m}{n} + \frac{m}{n}] \leq e^{-\frac{(m/n)^2}{3(\ell-1)m/n}} = e^{-m/(3n(\ell-1))} \leq \frac{1}{n^2}.$$

Similarly,

$$p'_\ell - p_\ell \leq \Pr_{A \in_R \left[ \binom{N \setminus \{\ell\}}{\ell} \right]} [w(A) < q - w_1] \leq n \Pr[B(m, \frac{\ell}{n}) < q - w_1].$$

We now need the lower bound on  $w_1$  given by Lemma A.4, which holds with probability  $1 - 1/n$ :

$$q - w_1 \leq \frac{\ell m}{n} - \left( \frac{m}{n} - \sqrt{\frac{4m \log n}{n}} \right) \leq \frac{\ell m}{n} - \frac{m}{2n},$$

the latter inequality following from  $m \geq 24n^2 \log n > 16n \log n$ . Assuming the lower bound on  $w_1$ ,

$$\Pr[\mathbb{B}(m, \frac{\ell}{n}) < q - w_1] \leq e^{-\frac{(m/(2n))^2}{3(\ell-1)m/n}} = e^{-m/(12n(\ell-1))} \leq \frac{1}{n^2}.$$

Therefore

$$p'_\ell - p_\ell \leq \left(1 - \frac{1}{n}\right) \cdot \frac{1}{n^2} + \frac{1}{n} \cdot 1 < \frac{2}{n}.$$

The formula for  $\mathbb{E}[\varphi_1]$  follows from Lemma A.6.  $\square$

It remains to relate  $p'_{\ell-1}$  and  $p'_\ell$ .

LEMMA A.8. *Suppose that  $m \geq 24n^3 \log n$ . Then*

$$\left| p'_\ell - \left( \frac{n}{2(n-\ell)} - \frac{\ell}{n-\ell} (1 - p'_{\ell-1}) \right) \right| \leq \frac{1}{n},$$

and so

$$-\frac{4}{n^2} \leq \mathbb{E}[\varphi_1] - \left( \frac{1}{2(n-\ell)} - \frac{\ell}{n(n-\ell)} + \frac{p'_{\ell-1}}{n-\ell} \right) \leq \frac{2}{n^2}.$$

PROOF. We have

$$\begin{aligned} p'_\ell &= \Pr_{A \in \mathcal{R}[\binom{N \setminus \{1\}}{\ell}]} [w(A) < q] \\ &= \frac{1}{\binom{n-1}{\ell}} \sum_{A \in \binom{N \setminus \{1\}}{\ell}} \Pr[w(A) < q] \\ &= \frac{1}{\binom{n-1}{\ell}} \sum_{A \in \binom{N}{\ell}} \Pr[w(A) < q] - \frac{1}{\binom{n-1}{\ell}} \sum_{A \in \binom{N \setminus \{1\}}{\ell-1}} \Pr[w(A) + w_1 < q] \\ &= \frac{n}{n-\ell} \Pr_{A \in \mathcal{R}[\binom{N}{\ell}]} \Pr[w(A) < q] - \frac{\ell}{n-\ell} \left( 1 - \Pr_{A \in \mathcal{R}[\binom{N \setminus \{1\}}{\ell-1}]} [w(A) + w_1 \geq q] \right) \\ &= \frac{n}{n-\ell} \Pr[\mathbb{B}(m, \frac{\ell}{n}) < q] - \frac{\ell}{n-\ell} (1 - p'_{\ell-1}), \end{aligned}$$

where the final equality follows from the second part of Lemma A.5. We proceed to estimate  $\Pr[\mathbb{B}(m, \frac{\ell}{n}) < q]$  using the Berry–Esseen theorem. The normalized binomial  $\mathbb{B}(m, \frac{\ell}{n}) - q$  is a sum of  $m$  independent copies of the random variable  $X$  with  $\Pr[X = 1 - \frac{\ell}{n}] = \frac{\ell}{n}$  and  $\Pr[X = -\frac{\ell}{n}] = 1 - \frac{\ell}{n}$ . The Berry–Esseen theorem states that

$$\left| \Pr[\mathbb{B}(m, \frac{\ell}{n}) - q < 0] - \Pr[\mathcal{N}(0, \sigma^2) < 0] \right| < \frac{\rho}{\sigma^3 \sqrt{m}},$$

where  $\sigma^2 = \mathbb{E}[X^2] = \frac{\ell}{n} (1 - \frac{\ell}{n})^2 + (1 - \frac{\ell}{n}) (\frac{\ell}{n})^2 = \frac{\ell}{n} (1 - \frac{\ell}{n})$  and  $\rho = \mathbb{E}[|X|^3] = \frac{\ell}{n} (1 - \frac{\ell}{n})^3 + (1 - \frac{\ell}{n}) (\frac{\ell}{n})^3 = \frac{\ell}{n} (1 - \frac{\ell}{n}) [(\frac{\ell}{n})^2 + (1 - \frac{\ell}{n})^2]$ . Since  $\Pr[\mathcal{N}(0, \sigma^2) < 0] = 1/2$ , we conclude that

$$\left| \Pr[\mathbb{B}(m, \frac{\ell}{n}) - q < 0] - \frac{1}{2} \right| < \frac{1}{\sqrt{m}} \frac{(\frac{\ell}{n})^2 + (1 - \frac{\ell}{n})^2}{\sqrt{\frac{\ell}{n} (1 - \frac{\ell}{n})}} \leq 2\sqrt{\frac{n}{m}},$$

since the denominator is at least  $\sqrt{\frac{1}{n} (1 - \frac{1}{n})}$ , and the numerator is at most  $2(1 - \frac{1}{n})^2 \leq 2\sqrt{1 - \frac{1}{n}}$ . Since  $m \geq 24n^3 \log n \geq 4n^3$ , we further have  $2\sqrt{\frac{n}{m}} \leq \frac{1}{n}$ .

The formula for  $\mathbb{E}[\varphi_1]$  follows from Lemma A.7.  $\square$

Lemma A.8 is simply a reformulation of Lemma 4.5.

#### A.4. Proof of Lemma 4.6

Let us recall Lemma 4.6.

LEMMA 4.6. *With probability  $1 - 2/n$ , we have that  $\sqrt{\frac{m \log n}{3n}} \leq \frac{m}{n} - w_1 \leq \sqrt{\frac{4m \log n}{n}}$ .*

We already proved the upper bound in Lemma A.4, using a simple union bound. The lower bound (corresponding to an upper bound on  $w_1$ ) is more difficult, because of the dependence between the individual bins. One way to overcome this difficulty is to use the Poisson approximation, given by the following theorem.

THEOREM A.9 ([MITZENMACHER AND UPFAL 2005]). *Let  $w_1, \dots, w_n$  be sampled according to the Balls and Bins distribution with  $m$  balls, and let  $X_1, \dots, X_n$  be  $n$  i.i.d. random variables sampled from the distribution  $\text{Pois}(\frac{m}{n})$ . Let  $f: \mathbb{R}^n \rightarrow \{0, 1\}$  be a boolean function over the weight vector, such that the probability  $p(w_1, \dots, w_n) = \Pr[f(w_1, \dots, w_n) = 1]$  is monotonically increasing or decreasing with the number of balls. Then  $p(w_1, \dots, w_n) \leq 2p(X_1, \dots, X_n)$ .*

The following lemma completes the proof of Lemma 4.6, since calculation shows that for all  $n \geq 1$ ,

$$\frac{m}{n} \sqrt{\frac{\log(n/\log(2n))}{m/n}} = \sqrt{\frac{m \log(n/\log(2n))}{n}} \geq \sqrt{\frac{m \log n}{3n}}.$$

(In fact, the minimum of  $\frac{\log(n/\log(2n))}{\log n}$  is obtained for  $n = 3$ , in which case it is roughly 0.47.)

LEMMA A.10. *Let  $\lambda = \frac{m}{n}$ . For any  $\varepsilon \leq \sqrt{\frac{\log(\frac{n}{\log(2n)})}{\lambda}}$ ,  $\Pr[w_1 > \lambda(1 - \varepsilon)] \leq \frac{1}{n}$ .*

PROOF. We define  $n$  i.i.d. random variables  $X_1, \dots, X_n$ , sampled from the distribution  $\text{Pois}(\lambda)$ . We first derive a concentration bound on  $\min_i X_i$ , after which we will make use of Theorem A.9 to obtain the desired result. By the definition of the Poisson distribution,

$$\Pr[\min_i X_i > t] = \Pr[X_1 > t]^n \leq \Pr[X_1 \neq t]^n \leq \left(1 - e^{-\lambda} \frac{\lambda^t}{t!}\right)^n \leq \left(1 - e^{-\lambda} \left(\frac{e\lambda}{t}\right)^t\right)^n.$$

The last inequality is due to the fact that  $t! \geq \left(\frac{t}{e}\right)^t$ , by Stirling's approximation. Setting  $t = (1 - \varepsilon)\lambda$ , we get

$$\begin{aligned} \Pr[\min_i X_i > (1 - \varepsilon)\lambda] &\leq \left(1 - e^{-\lambda} \left(\frac{e\lambda}{(1 - \varepsilon)\lambda}\right)^{(1 - \varepsilon)\lambda}\right)^n \\ &= \left(1 - e^{-\lambda} \left(\frac{e}{1 - \varepsilon}\right)^{(1 - \varepsilon)\lambda}\right)^n \\ &\leq \left(1 - e^{-\varepsilon\lambda} e^{(1 - \varepsilon)\varepsilon\lambda}\right)^n \\ &= \left(1 - e^{-\varepsilon^2\lambda}\right)^n \leq e^{-ne^{-\varepsilon^2\lambda}}. \end{aligned}$$

The second inequality follows from the inequality  $\frac{1}{1-x} \geq e^x$ , for  $|x| < 1$ . The third inequality follows from the inequality  $1 - x \leq e^{-x}$ .



Now, for any  $\varepsilon \leq \sqrt{\frac{\log\left(\frac{n}{\log(2n)}\right)}{\lambda}}$ , we have

$$e^{-ne^{-\varepsilon^2\lambda}} \leq e^{-ne^{-\log\left(\frac{n}{\log(2n)}\right)}} = e^{-\log(2n)} = \frac{1}{2n}.$$

A simple coupling argument shows that  $\Pr[\min_i w_i > (1-\varepsilon)\lambda]$  is monotone increasing in the number of balls (here,  $f(w_1, \dots, w_n)$  is 1 if and only if  $\min_i w_i > (1-\varepsilon)\lambda$ ). Therefore Theorem A.9 holds, and we have

$$\Pr[\min_i w_i > (1-\varepsilon)\lambda] \leq 2\Pr[\min_i X_i > (1-\varepsilon)\lambda] \leq \frac{1}{n},$$

which concludes the proof.  $\square$

#### A.5. Proof of Lemma 4.7

LEMMA 4.7. *Suppose  $q = \ell \frac{m}{n}$  for an integer  $\ell$  satisfying  $\gamma \leq \frac{\ell-1}{n} \leq 1-\gamma$ , and let*

$$t_\varepsilon = \Pr_{A \in \mathcal{R}^{\left[\begin{smallmatrix} N \\ \ell-1 \end{smallmatrix} \right]}} \left[ w(A) + w_1 \geq q : w_1 = \frac{m}{n} - \varepsilon \sqrt{\frac{m \log n}{n}} \right].$$

Then for  $m \geq 4n^3$ ,

$$t_\varepsilon \geq \frac{1}{2} - \frac{\varepsilon}{2\pi\gamma} \sqrt{\frac{\log n}{n}} - \frac{1}{n}.$$

PROOF. The idea of the proof is to replace  $w(A)$  by the weight of a random set of size  $\ell-1$ . A simple coupling argument shows that

$$\begin{aligned} t_\varepsilon &\geq \Pr_{A \in \mathcal{R}^{\left[\begin{smallmatrix} N \\ \ell-1 \end{smallmatrix} \right]}} \left[ w(A) + w_1 \geq q : w_1 = \frac{m}{n} - \varepsilon \sqrt{\frac{m \log n}{n}} \right] \\ &= \Pr \left[ \text{B}\left(m, \frac{\ell-1}{n}\right) \geq \frac{(\ell-1)m}{n} + \varepsilon \sqrt{\frac{m \log n}{n}} \right], \end{aligned}$$

using the second part of Lemma A.5.

As in the proof of Lemma A.8, since  $m \geq 4n^3$ , we can use the Berry–Esseen theorem to estimate the latter expression up to an additive error of  $\frac{1}{n}$ :

$$\begin{aligned} t_\varepsilon &\geq \Pr \left[ \text{B}\left(m, \frac{\ell-1}{n}\right) \geq \frac{(\ell-1)m}{n} + \varepsilon \sqrt{\frac{m \log n}{n}} \right] \\ &\geq \Pr \left[ \mathcal{N}\left(\frac{(\ell-1)m}{n}, \frac{(\ell-1)m}{n} \left(1 - \frac{\ell-1}{n}\right)\right) \geq \frac{(\ell-1)m}{n} + \varepsilon \sqrt{\frac{m \log n}{n}} \right] - \frac{1}{n}. \end{aligned}$$

In order to estimate the latter probability, we use the bound  $\Pr[\mathcal{N}(0, 1) \geq x] \geq 1/2 - \frac{x}{\sqrt{2\pi}}$  (for  $x \geq 0$ ), which follows from  $\Pr[\mathcal{N}(0, 1) \geq 0] = 1/2$  and the fact that the density of  $\mathcal{N}(0, 1)$  is bounded by  $1/\sqrt{2\pi}$ . In our case,

$$x = \varepsilon \sqrt{\frac{m \log n}{n}} / \sqrt{\frac{(\ell-1)m}{n} \left(1 - \frac{\ell-1}{n}\right)} \leq \varepsilon \sqrt{\frac{m \log n}{n}} / \sqrt{\gamma^2 m} = \varepsilon \sqrt{\frac{\log n}{\gamma^2 n}}.$$

Therefore

$$t_\varepsilon \geq \frac{1}{2} - \frac{\varepsilon}{2\pi\gamma} \sqrt{\frac{\log n}{n}} - \frac{1}{n}.$$

□

## B. MISSING PROOFS FROM SECTION 5

### B.1. Proof of Lemma 5.2

LEMMA 5.2. *Assume that  $m$  voters submit the votes according to the exponential distribution over candidates in which the probability the candidate  $i$  is voted for by each voter is proportional to  $\rho^{-i}$ , for some  $0 < \rho < 1/2$ . There is a (universal) constant  $C > 0$  such that if  $m \geq C\rho^{-n}(1-2\rho)^{-2} \log n$  then the resulting weight vector is super-increasing with probability  $1 - O(\frac{1}{n})$ . Furthermore, as  $m \rightarrow \infty$ , the probability approaches 1.*

PROOF. The proof uses Bernstein's inequality with a subsequent application of the union bound. Consider a sequence  $w_1 \leq w_2 \leq \dots \leq w_n$ . The sequence is clearly super-increasing if for every  $i = 2, \dots, n$ ,  $w_i/w_{i-1} \geq 2$ , and  $w_1 > 0$ . We now lower bound the probability of this event, by upper-bounding the probability of the following bad events:  $E_i$  is the event that  $w_i < 2w_{i-1}$  (for  $i = 2, \dots, n$ ), and  $E_1$  is the event that  $w_1 = 0$ . A union bound shows that the sequence  $\mathbf{w}$  is super-increasing with probability at least  $1 - \sum_{i=1}^n \Pr[E_i]$ .

First note that the probability that voter  $j$  votes for candidate  $i$  is equal to

$$p_i = \frac{\rho^{n-i}}{\sum_{i=1}^n \rho^{n-i}} = \frac{\rho^{n-i}(1-\rho)}{1-\rho^n} = \Theta(\rho^{n-i}).$$

Bounding the probability of  $E_1$  is easy:

$$\Pr[E_1] = (1-p_1)^m \leq e^{-p_1 m} = e^{-\Theta(\rho^{n-1}m)}.$$

In order to bound the probability of  $E_i$  for  $i \neq 1$ , consider the random variable  $X = 2w_{i-1} - w_i$ . This random variable is a sum of  $m$  i.i.d. random variables  $X^{(1)}, \dots, X^{(m)}$  corresponding to the different voters with the following distribution:

$$X^{(j)} = \begin{cases} 2 & \text{w.p. } p_{i-1}, \\ -1 & \text{w.p. } p_i, \\ 0 & \text{w.p. } 1 - p_{i-1} - p_i. \end{cases}$$

Using the identity  $p_{i-1} = \rho p_i$ , the moments of  $X$  are

$$\mathbb{E}[X] = m \mathbb{E}[X^{(j)}] = (2\rho - 1)p_i m = \Theta((2\rho - 1)\rho^{n-i}m),$$

$$\text{Var}[X] = m(\mathbb{E}[X^{(j)2}] - \mathbb{E}[X^{(j)}]^2) = (4\rho + 1)p_i m - (2\rho - 1)^2 p_i^2 m = O(\rho^{n-i}m).$$

Since  $|X^{(j)} - \mathbb{E}[X^{(j)}]| = O(1)$ , Bernstein's equality gives

$$\begin{aligned} \Pr[E_i] &= \Pr[X > 0] \\ &\leq \exp - \frac{\frac{1}{2} \mathbb{E}[X]^2}{\text{Var}[X] + O(\mathbb{E}[X])} \\ &= \exp - \frac{\Theta((2\rho - 1)^2 \rho^{2(n-i)} m^2)}{O(\rho^{n-i}m)} \\ &= \exp - \Omega((2\rho - 1)^2 \rho^{n-i}m). \end{aligned}$$

Summarizing,

$$\sum_{i=1}^n \Pr[E_i] \leq e^{-\Theta(\rho^{n-1}m)} + \sum_{i=2}^n e^{-\Omega((2\rho-1)^2 \rho^{n-i}m)}.$$

When  $m \geq C\rho^{-n}(2\rho-1)^{-2} \log n$  for an appropriate  $C$ , all the terms are  $O(1/n^2)$ , and so the total error probability is  $O(1/n)$ , proving the first part of lemma. As  $m \rightarrow \infty$ , all the terms tend to 0, and so the total error probability tends to 0, proving the second part of the lemma. □

## B.2. Proof of Lemma 5.4

LEMMA 5.4. Suppose that  $\mathbf{p} = (p_1, \dots, p_n)$  is a SI sequence summing to 1, and let  $w_1, \dots, w_n$  be obtained by sampling  $m$  times from the distribution  $p_1, \dots, p_n$ .

Suppose that  $T \in (0, 1]$ , say  $T \in I^{\mathbf{p}}(P)$  for some  $P \subseteq \{1, \dots, n\}$ . If the distance of  $T$  from the endpoints  $\tilde{p}(P), \tilde{p}(P^+)$  of  $I^{\mathbf{p}}(P)$  is at least  $\Delta = \sqrt{\log(nm)/m}$  then with probability  $1 - \frac{2}{(nm)^2}$  it holds that if  $\mathbf{w}$  is SI then for all  $i \in N$ ,  $\varphi_i^{\mathbf{w}}(mT) = \varphi_i^{\mathbf{p}}(T)$ .

PROOF. Suppose that  $\mathbf{w}$  is super-increasing. Lemma 6.1 implies that  $I^{\mathbf{w}}(P) = (\tilde{w}(P), \tilde{w}(P^+)]$ , since both  $\mathbf{p}$  and  $\mathbf{w}$  are super-increasing (a priori, it could be that  $P^+$  would have different values when defined with respect to  $\mathbf{p}$  and to  $\mathbf{w}$ , but this is ruled out by Lemma 6.1). The idea of the proof is to show that with high probability,  $mT \in I^{\mathbf{w}}(P)$ , and then the lemma follows from Proposition 5.3. We do that by upper-bounding the probability of the following two bad events:  $\tilde{w}(P) \geq mT$  and  $\tilde{w}(P^+) < mT$ .

The random variable  $\tilde{w}(P)$  is a sum of  $m$  i.i.d. indicator random variables which are 1 with probability  $\tilde{p}(P)$ . Therefore  $\mathbb{E}[\tilde{w}(P)] = m\tilde{p}(P)$ . Hoeffding's inequality shows that

$$\Pr[\tilde{w}(P) \geq mT] \leq \Pr[\tilde{w}(P) \geq \mathbb{E}[\tilde{w}(P)] + m\Delta] \leq e^{-2\Delta^2 m}.$$

Similarly  $\Pr[\tilde{w}(P^+) < mT] \leq e^{-2\Delta^2 m}$ . When  $\Delta \geq \sqrt{\log n/m}$ , both error probabilities are at most  $1/(nm)^2$ .  $\square$

## C. MISSING PROOFS FROM SECTION 6

### C.1. A Greedy Algorithm for Finding $A(q)$

Given a point  $q \in (0, w(N)]$  and a vector of super-increasing weights  $\mathbf{w}$ , it is possible to find  $A(q)$  in time  $O(n)$ .

LEMMA C.1. Algorithm 1 calculates  $A(q)$  in linear time.

PROOF. As stated, the algorithm does not in fact run in linear time, but it is easy to modify it so that it does run in linear time. It remains to prove that it calculates  $A(q)$  correctly.

Let  $A(q) = a_0, \dots, a_r$ , so that  $A(q)^- = a_0, \dots, a_{r-1}, a_r+1, \dots, n$ . Denote by  $A_i$  the value of  $A$  in the algorithm after  $i$  iterations of the loop. We prove by induction on  $i$  that  $A_i = A(q) \cap \{1, \dots, i\}$ , which shows that the algorithm returns  $A(q)$ .

The inductive claim trivially holds for  $i = 0$ . Assuming that  $A_{i-1} = A(q) \cap \{1, \dots, i-1\}$ , we now prove that  $A_i = A(q) \cap \{1, \dots, i\}$ . We consider two cases:  $i \notin A(q)$  and  $i \in A(q)$ . If  $i \notin A(q)$  then  $q \leq w(A(q)) = w(A_{i-1}) + w(A(q) \cap \{i, \dots, n\}) \leq w(A_{i-1}) + w(\{i+1, \dots, n\})$ , and so  $i$  is not added to  $A_i$ . Suppose now that  $i \in A(q)$ . If  $a_r = i$  then  $q > w(A(q)^-) = w(A_{i-1}) + w(\{i+1, \dots, n\})$ , and so  $i$  is added to  $A_i$ . If  $a_r > i$  then  $q > w(A(q)^-) \geq w(A_{i-1}) + w_i > w(A_{i-1}) + w(\{i+1, \dots, n\})$ , since  $\mathbf{w}$  is super-increasing, and so  $i$  is added to  $A_i$  in this case as well.  $\square$

---

**ALGORITHM 1:** Algorithm Find-Set for finding  $A(q)$

---

```

Input:  $w, q$ 
 $A \leftarrow \emptyset$ 
for  $i \leftarrow 1$  to  $n$  do
    if  $q > w(A \cup \{i+1, \dots, n\})$  then
         $A \leftarrow A \cup \{i\}$ 
    end
end
return  $A$ 

```

---

### C.2. Finding $A(q)$ when Weights are $w_i = d^{n-i}$

LEMMA C.2. Suppose  $w_i = d^{n-i}$  for some integer  $d \geq 2$ , and let  $q \in (0, w(N)]$ . Write  $\lceil q \rceil$  in base  $d$ :  $\lceil q \rceil = (t_1 \dots t_n)_d$ . If the base  $d$  representation only consists of the digits 0 and 1 then  $A(q) = \{i \in N : t_i = 1\}$ . Otherwise, let  $\ell$  be the minimal index such that  $t_\ell > 1$ , and let  $k < \ell$  be the maximal index less than  $\ell$  satisfying  $t_k = 0$  (the proof shows that such an index exists). Then  $A(q) = \{i \in \{1, \dots, k-1\} : t_i = 1\} \cup \{k\}$ .

PROOF. Suppose first that  $t_i \in \{0, 1\}$  for all  $i \in N$ , and let  $Q(q) = \{i \in N : t_i = 1\}$ . Since  $\lceil q \rceil \geq 1$ ,  $Q(q) \neq \emptyset$ . Lemma 6.1 shows that  $w(Q(q)^-) < w(Q(q))$  and so  $q = w(Q(q)) \in (w(Q(q)^-), w(Q(q))]$ , showing that  $A(q) = Q(q)$ .

Suppose next that  $\ell$  is the minimal index such that  $t_\ell > 1$ . If  $t_k = 1$  for all  $k < \ell$  then

$$q > \lceil q \rceil - 1 \geq \sum_{j=1}^{\ell-1} w_j + 2w_\ell - 1 \geq w(N),$$

since the fact that the  $w_i$  are integral and super-increasing implies that

$$w_\ell \geq \sum_{j=\ell+1}^n w_j + 1.$$

We conclude that the maximal index  $k < \ell$  satisfying  $t_k = 0$  exists. Let  $Q(q) = \{i \in \{1, \dots, k-1\} : t_i = 1\} \cup \{k\}$ . On the one hand,

$$q \leq \lceil q \rceil \leq \sum_{j \in Q(q) \setminus \{k\}} w_j + (d-1) \sum_{j=k+1}^n w_j < w(Q(q)).$$

On the other hand,

$$\begin{aligned} q > \lceil q \rceil - 1 &\geq \sum_{j \in Q(q) \setminus \{k\}} w_j + \sum_{j=k+1}^{\ell-1} w_j + 2w_\ell - 1 \\ &\geq \sum_{j \in Q(q) \setminus \{k\}} w_j + \sum_{j=k+1}^n w_j = w(Q(q)^-). \end{aligned}$$

Therefore  $A(q) = Q(q)$ .  $\square$

### C.3. Proof of Theorem 6.3

First, we recall the statement of Theorem 6.3.

THEOREM 6.3. Consider an agent  $i \in N$  and a prescribed quota value  $q \in (0, w(N)]$ . Let  $A(q) = \{a_0, \dots, a_r\}$ . If  $i \notin A(q)$  then

$$\varphi_i(q) = \sum_{\substack{t \in \{0, \dots, r\} \\ a_t > i}} \frac{1}{a_t \binom{a_t-1}{t}}.$$

If  $i \in A(q)$ , say  $i = a_s$ , then

$$\varphi_i(q) = \frac{1}{a_s \binom{a_s-1}{s}} - \sum_{t > s} \frac{1}{a_t \binom{a_t-1}{t-1}}.$$

PROOF. Lemma 6.2 shows that  $\varphi_i^w(q) = \varphi_i^{\mathbf{b}}(\beta(A(q)))$ , where  $\mathbf{b} = 2^{n-1}, \dots, 1$ . Therefore we can assume without loss generality that  $\mathbf{w} = 2^{n-1}, \dots, 1$ , i.e.,  $w_i = 2^{n-i}$ , and that  $q = \sum_{j \in A(q)} w_j$ .

Recall that  $\varphi_i(q)$  is the probability that  $w(P_i(\pi)) \in [q - w_i, q)$ , where  $\pi$  is chosen randomly from  $\text{Sym}_n$ , and  $P_i(\pi)$  is the set of predecessors of  $i$  in  $\pi$ . The idea of the proof is to consider the maximal  $\tau \in \{1, \dots, r + 1\}$  such that  $a_t \in P_i(\pi)$  for all  $t < \tau$ . We will show that when  $i \notin A(q)$ , each possible value of  $\tau(\pi)$  corresponds to one summand in the expression for  $\varphi_i(q)$ . When  $i \in A(q)$ , say  $i = a_s$ , we will show that the events that  $i$  is pivotal with respect to  $q$  and that  $i$  is pivotal with respect to  $q - w_i$  are disjoint, and their union is an event having probability  $1/a_s \binom{a_s - 1}{s}$ .

Suppose that  $i$  is pivotal for  $\pi$  and  $\tau(\pi) = \tau$ . We start by showing that  $\tau \leq r$ , ruling out the case  $\tau = r + 1$ . If  $\tau = r + 1$  then by definition

$$w(P_i(\pi)) \geq \sum_{j \in A(q)} w_j = q,$$

contradicting the assumption  $w(P_i(\pi)) < q$ . Therefore  $\tau \leq r$ , and so  $a_\tau$  is well-defined. We claim that if  $k \in P_i(\pi)$  for some agent  $k < a_\tau$  then  $k \in A(q)$ . Indeed, otherwise

$$w(P_i(\pi)) \geq \sum_{t=0}^{\tau-1} w_{a_t} + w_k \geq \sum_{t=0}^{\tau-1} w_{a_t} + w_{a_{\tau-1}} > \sum_{t=0}^{\tau-1} w_{a_t} + \sum_{j=a_\tau}^n w_j \geq q,$$

again contradicting  $w(P_i(\pi)) < q$  (the third inequality made use of the fact that  $w$  is super-increasing).

Furthermore, we claim that  $a_\tau \geq i$ . Otherwise,

$$w(P_i(\pi)) \leq \sum_{t=0}^{\tau-1} w_{a_t} + \sum_{j=a_\tau+1}^n w_j - w_i < \sum_{t=0}^{\tau} w_{a_t} - w_i \leq q - w_i,$$

contradicting the assumption  $w(P_i(\pi)) \geq q - w_i$ .

Summarizing, we have shown that  $\tau \leq r$ ,  $a_\tau \geq i$  and

$$P_i(\pi) \cap \{1, \dots, a_\tau\} = \{a_0, \dots, a_{\tau-1}\}. \quad (2)$$

Denote this event  $E_\tau$ , and call a  $\tau \leq r$  satisfying  $a_\tau \geq i$  *legal*.

Suppose first that  $i \notin A(q)$ . We have shown above that if  $i$  is pivotal then  $E_\tau$  happens for some legal  $\tau$ . We claim that the converse is also true. Indeed, given  $E_\tau$  defined with respect to a permutation  $\pi$ , and for some legal  $\tau$ , the weight of  $P_i(\pi)$  can be bounded as follows.

$$\sum_{t=0}^{\tau-1} w_{a_t} \leq w(P_i(\pi)) \leq \sum_{t=0}^{\tau-1} w_{a_t} + \sum_{j=a_\tau+1}^n w_j < \sum_{t=0}^{\tau} w_{a_t}.$$

The second inequality follows from the definition of  $\tau$ , whereas the third inequality follows as before from the definition of a super-increasing sequence. The upper bound is clearly at most  $q$ , and the lower bound satisfies

$$\sum_{t=0}^{\tau-1} w_{a_t} \geq q - \sum_{j=a_\tau}^n w_j > q - w_{a_{\tau-1}} \geq q - w_i,$$

since  $i < a_\tau$ .

It remains to calculate  $\Pr[E_\tau]$ . The event  $E_\tau$  states that the restriction of  $\pi$  to  $\{1, \dots, a_\tau\}$  consists of the elements  $\{a_0, \dots, a_{\tau-1}\}$  in some order, followed by  $i$  (recall that  $i \leq a_\tau$ ). For each of the  $\tau!$  possible orders, the probability of this is  $1/a_\tau \cdots (a_\tau - \tau) = (a_\tau - \tau - 1)!/a_\tau!$ , and so

$$\Pr[E_\tau] = \frac{\tau!(a_\tau - \tau - 1)!}{a_\tau!} = \frac{1}{a_\tau \binom{a_\tau - 1}{\tau}}. \quad (3)$$

Summing over all legal  $\tau$ , we obtain the formula in the statement of the theorem. This completes the proof in the case  $i \notin A(q)$ .

Suppose next that  $i \in A(q)$ , say  $i = a_s$ . Since  $a_\tau \geq a_s = i$  while  $i \notin P_i(\pi)$ , we deduce that  $\tau = s$ . Therefore the event  $E_s$  happens. Conversely, when  $E_s$  happens,

$$w(P_i(\pi)) \leq \sum_{t=0}^{s-1} w_{a_t} + \sum_{j=a_s+1}^n w_j < \sum_{t=0}^s w_{a_t} \leq q.$$

Therefore  $i$  is pivotal (with respect to  $q$ ) if and only if  $E_s$  happens and  $w(P_i(\pi)) \geq q - w_i$ .

It is easy to check that  $A(q - w_i) = A(q) \setminus \{i\} = a_0, \dots, a_{s-1}, a_{s+1}, \dots, a_r$ . The argument above shows that if  $i$  is pivotal with respect to  $q - w_i$  then for some  $\tau' \geq s + 1$ ,

$$P_i(\pi) \cap \{1, \dots, a_{\tau'}\} = \{a_0, \dots, a_{s-1}, a_{s+1}, \dots, a_{\tau'-1}\}.$$

In particular, the event  $E_s$  happens. Conversely, when  $E_s$  happens,

$$w(P_i(\pi)) \geq \sum_{t=0}^{s-1} w_{a_t} \geq q - w_{a_s} - \sum_{j=a_s+1}^n w_j > (q - w_{a_s}) - w_{a_s}.$$

Therefore  $i$  is pivotal with respect to  $q - w_i$  if and only if  $E_s$  happens and  $w(P_i(\pi)) < q - w_i$ . We conclude that

$$\Pr[w_i \text{ is pivotal with respect to } q] = \Pr[E_s] - \Pr[w_i \text{ is pivotal with respect to } q - w_i].$$

Above we have calculated  $\Pr[E_s] = 1/a_s \binom{a_s-1}{s}$ , and we obtain the formula in the statement of the theorem.  $\square$

#### C.4. Proof of Lemma 6.6

To prove Lemma 6.6, we will need some combinatorial identities.

LEMMA C.3. *Let  $p, t$  be integers satisfying  $p > t \geq 1$ . Then*

$$\frac{1}{p \binom{p-1}{t}} + \frac{1}{p \binom{p-1}{t-1}} = \frac{1}{(p-1) \binom{p-2}{t-1}}.$$

PROOF. The proof is a simple calculation:

$$\begin{aligned} \frac{1}{p \binom{p-1}{t}} + \frac{1}{p \binom{p-1}{t-1}} &= \frac{t!(p-t-1)! + (t-1)!(p-t)!}{p!} \\ &= \frac{(t-1)!(p-t-1)! [t + (p-t)]}{p!} = \frac{(t-1)!(p-t-1)!}{(p-1)!} = \frac{1}{(p-1) \binom{p-2}{t-1}}. \end{aligned}$$

$\square$

LEMMA C.4. *Let  $p, t, k$  be integers satisfying  $p > t \geq 0$  and  $k \geq 0$ . Then*

$$\frac{1}{p \binom{p-1}{t}} - \sum_{\ell=1}^k \frac{1}{(p+\ell) \binom{p+\ell-1}{t+\ell-1}} = \frac{1}{(p+k) \binom{p+k-1}{t+k}}.$$

In particular,

$$\frac{1}{p \binom{p-1}{t}} = \sum_{\ell=1}^{\infty} \frac{1}{(p+\ell) \binom{p+\ell-1}{t+\ell-1}}.$$

PROOF. The proof is by induction on  $k$ . If  $k = 0$  then there is nothing to prove. For  $k > 0$  we have

$$\frac{1}{p \binom{p-1}{t}} - \sum_{\ell=1}^k \frac{1}{(p+\ell) \binom{p+\ell-1}{t+\ell-1}} = \frac{1}{(p+k-1) \binom{p+k-2}{t+k-1}} - \frac{1}{(p+k) \binom{p+k-1}{t+k}} = \frac{1}{(p+k) \binom{p+k-1}{t+k}},$$

using Lemma C.3. The second expression of the lemma follows from rearranging the first formula and taking the limit  $k \rightarrow \infty$ .  $\square$

We are now ready to prove Lemma 6.6. First, recall the statement of the lemma.

**LEMMA 6.6.** *Given a quota  $q \in (0, w(N)]$ , let  $A(q) = \{a_0, \dots, a_r\}$ . Given some  $i \in N \setminus \{n\}$ , (a) if  $i, i+1 \in A(q)$  or  $i, i+1 \notin A(q)$  then  $\varphi_i(q) = \varphi_{i+1}(q)$ ; (b) if  $i \notin A(q)$  and  $i+1 \in A(q)$  then  $\varphi_i(q) \geq \varphi_{i+1}(q)$ , with equality if and only if  $i+1 = a_r$ ; (c) if  $i \in A(q)$  and  $i+1 \notin A(q)$  then  $\varphi_i(q) > \varphi_{i+1}(q)$ .*

**PROOF.** We write  $A(q) = \{a_1, \dots, a_r\}$ . First, let us assume that neither  $i$  nor  $i+1$  are in  $A(q)$ . For any  $t \in \{0, \dots, r\}$ ,  $a_t > i$  if and only if  $a_t > i+1$ . Employing the formula used in Theorem 6.3, we have that

$$\begin{aligned} \varphi_i(q) &= \sum_{\substack{t \in \{0, \dots, r\}: \\ a_t > i}} \frac{1}{a_t \binom{a_t-1}{t}} \\ &= \sum_{\substack{t \in \{0, \dots, r\}: \\ a_t > i+1}} \frac{1}{a_t \binom{a_t-1}{t}} = \varphi_{i+1}(q). \end{aligned}$$

Next, if  $i, i+1 \in A(q)$  then there is some  $s$  such that  $i = a_s$  and  $i+1 = a_{s+1}$ . Therefore,

$$\begin{aligned} \varphi_i(q) &= \frac{1}{a_s \binom{a_s-1}{s}} - \sum_{\substack{t \in \{0, \dots, r\}: \\ a_t > i}} \frac{1}{a_t \binom{a_t-1}{t-1}} = \frac{1}{a_s \binom{a_s-1}{s}} - \frac{1}{a_{s+1} \binom{a_{s+1}-1}{s}} - \sum_{\substack{t \in \{0, \dots, r\}: \\ a_t > i+1}} \frac{1}{a_t \binom{a_t-1}{t-1}} \\ &= \frac{1}{a_s \binom{a_s-1}{s}} - \frac{1}{(a_s+1) \binom{a_s}{s}} - \sum_{\substack{t \in \{0, \dots, r\}: \\ a_t > i+1}} \frac{1}{a_t \binom{a_t-1}{t-1}} \end{aligned} \quad (4)$$

$$= \frac{1}{(a_s+1) \binom{a_s}{s+1}} - \sum_{\substack{t \in \{0, \dots, r\}: \\ a_t > i+1}} \frac{1}{a_t \binom{a_t-1}{t-1}} \quad (5)$$

$$= \frac{1}{(a_{s+1}) \binom{a_{s+1}-1}{s+1}} - \sum_{\substack{t \in \{0, \dots, r\}: \\ a_t > i+1}} \frac{1}{a_t \binom{a_t-1}{t-1}} = \varphi_{i+1}(q)$$

The transition from (4) to (5) is due to Lemma C.3, and the last equality uses Theorem 6.3.

Now, suppose that  $i \notin A(q)$  but  $i+1 \in A(q)$ ; writing  $i+1 = a_s$ , we have that

$$\begin{aligned} \varphi_i(q) - \varphi_{i+1}(q) &= \sum_{t=s}^r \frac{1}{a_t \binom{a_t-1}{t}} - \left[ \frac{1}{a_s \binom{a_s-1}{s}} - \sum_{t=s+1}^r \frac{1}{a_t \binom{a_t-1}{t-1}} \right] \\ &= \sum_{t=s+1}^r \left[ \frac{1}{a_t \binom{a_t-1}{t}} + \frac{1}{a_t \binom{a_t-1}{t-1}} \right] = \sum_{t=s+1}^r \frac{1}{(a_t-1) \binom{a_t-2}{t-1}}, \end{aligned}$$

using Lemma C.3 again. Therefore,  $\varphi_i(q) \geq \varphi_{i+1}(q)$ , with equality if and only if  $s = r$ .

Finally, suppose that  $i \in A(q)$  and  $i+1 \notin A(q)$ . According to Theorem 6.3,  $\varphi_i(q) = \frac{1}{a_s \binom{a_s-1}{s}} - \sum_{t=s+1}^r \frac{1}{a_t \binom{a_t-1}{t-1}}$ , and  $\varphi_{i+1}(q) = \sum_{a_t > i+1} \frac{1}{a_t \binom{a_t-1}{t}}$ . If  $s = r$ , then the claim trivially holds. Since

$i + 1 \notin A(q)$ , it must be that  $a_{s+1} \geq a_s + 2$ , and in general  $a_{s+\ell} \geq a_s + \ell + 1$ . First, we note that

$$\varphi_{i+1}(q) = \sum_{a_t > i+1} \frac{1}{a_t \binom{a_t-1}{t}} \leq \sum_{t=s+1}^r \frac{1}{a_t \binom{a_t-1}{t}};$$

therefore:

$$\begin{aligned} \varphi_i(q) - \varphi_{i+1}(q) &\geq \frac{1}{a_s \binom{a_s-1}{s}} - \sum_{t=s+1}^r \left( \frac{1}{a_t \binom{a_t-1}{t-1}} + \frac{1}{a_t \binom{a_t-1}{t}} \right) \\ &= \frac{1}{a_s \binom{a_s-1}{s}} - \sum_{t=s+1}^r \frac{1}{(a_t - 1) \binom{a_t-2}{t-1}} \\ &= \frac{1}{a_s \binom{a_s-1}{s}} - \sum_{\ell=1}^{r-s} \frac{1}{(a_{s+\ell} - 1) \binom{a_{s+\ell}-2}{s+\ell-1}} \end{aligned} \quad (6)$$

$$\geq \frac{1}{a_s \binom{a_s-1}{s}} - \sum_{\ell=1}^{r-s} \frac{1}{(a_s + \ell - 1) \binom{a_s+\ell-2}{s+\ell-1}} \quad (7)$$

$$= \frac{1}{(a_s + r - s) \binom{a_s+r-s-1}{r}} > 0 \quad (8)$$

Transition (6) uses Lemma C.3; transition (7) uses the fact that  $\frac{1}{(m-1) \binom{m-2}{k}}$  is monotone decreasing in  $m$ , along with the fact that  $a_{s+\ell} \geq a_s + \ell + 1$ ; finally, (8) uses Lemma C.4.  $\square$

### C.5. Proof of Lemma 6.9

First, let us recall the statement of Lemma 6.9.

**LEMMA 6.9.** *For any  $S \subseteq N$ , and any  $i \in N$ , if  $i \notin S^-$  then  $\varphi_i(w(S^-)) < \varphi_i(w(S))$ . If  $i \in S^-$  then  $\varphi_i(w(S^-)) > \varphi_i(w(S))$ .*

*Moreover,  $|\varphi_i(w(S)) - \varphi_i(w(S^-))| = \frac{1}{n}$  if one of the following holds: (a)  $S = \{n\}$ ; (b)  $i < n$  and  $S = \{1, \dots, i\}$  or  $S = \{i, n\}$ ; or (c)  $i = n$  and  $S = \{n-1\}$ . Otherwise,  $|\varphi_i(w(S)) - \varphi_i(w(S^-))| \leq \frac{1}{n(n-1)}$ .*

**PROOF.** Given a non-empty set  $S \subseteq N$ , we define  $\varphi_+ = \varphi_i(w(S))$  and  $\varphi_- = \varphi_i(w(S^-))$ . Let  $S = a_0, \dots, a_r$ . We have  $S^- = a_0, \dots, a_{r-1}, a_r + 1, \dots, n$ .

Suppose first that  $i > a_r$ , and let  $s$  be the index of  $i$  in the sequence  $S^-$ . According to Theorem 6.3,  $\varphi_+ = 0$  and

$$\varphi_- = \frac{1}{i \binom{i-1}{s}} - \sum_{\ell=1}^{n-i} \frac{1}{(i+\ell) \binom{i+\ell-1}{s+\ell-1}} = \frac{1}{n \binom{n-1}{s+n-i}}.$$

It is immediately obvious that  $\varphi_- > \varphi_+$ ; furthermore,  $|\varphi_+ - \varphi_-| \leq \frac{1}{n(n-1)}$ , unless  $s + n - i \in \{0, n-1\}$ . If  $s + n - i = 0$  then  $s = 0$  and  $i = n$ , implying  $S^- = \{n\}$  and so  $S = \{n-1\}$ . If  $s + n - i = n - 1$  then  $s = i - 1$  and so  $S^- = \{1, \dots, n\}$ , which is impossible.

Suppose next that  $i = a_r$ . Since  $i = a_r$ , it must be the case that  $i \notin S^-$ . Moreover, according to Theorem 6.3,

$$\varphi_+ - \varphi_- = \frac{1}{i \binom{i-1}{r}} - \sum_{\ell=1}^{n-i} \frac{1}{(i+\ell) \binom{i+\ell-1}{r+\ell-1}} = \frac{1}{n \binom{n-1}{r+n-i}}.$$

thus  $\varphi_+ > \varphi_-$ . Furthermore,  $|\varphi_+ - \varphi_-| \leq \frac{1}{n(n-1)}$  unless  $r + n - i \in \{0, n-1\}$ . If  $r + n - i = 0$  then  $r = 0$  and  $i = n$ , and so  $S = \{n\}$ . If  $r + n - i = n$  then  $r = i - 1$  and so  $S = 1, \dots, i$ .



Finally, suppose that  $i < a_r$ . If  $i \notin S$  then it cannot be the case that  $i \in S^-$ . Moreover,

$$\varphi_+ - \varphi_- = \frac{1}{a_r \binom{a_r-1}{r}} - \sum_{\ell=1}^{n-a_r} \frac{1}{(a_r + \ell) \binom{a_r+\ell-1}{r+\ell-1}} = \frac{1}{n \binom{n-1}{r+n-a_r}}.$$

so  $\varphi_+ > \varphi_-$ . Furthermore,  $|\varphi_+ - \varphi_-| \leq \frac{1}{n \binom{n-1}{r+n-a_r}}$  unless  $r+n-a_r \in \{0, n-1\}$ . If  $r+n-a_r = 0$  then  $r = 0$  and  $a_r = n$ , and so  $S = \{n\}$ . If  $r+n-a_r = n-1$  then  $a_r = r+1$ , which implies  $S = \{1, \dots, r+1\}$ . However, this contradicts the assumption that  $i \notin S$ .

If  $i < a_r$  and  $i \in S$  then  $i$  must be in  $S^-$  as well. Therefore,

$$\varphi_- - \varphi_+ = \frac{1}{a_r \binom{a_r-1}{r-1}} - \sum_{\ell=1}^{n-a_r} \frac{1}{(a_r + \ell) \binom{a_r+\ell-1}{r+\ell-2}} = \frac{1}{n \binom{n-1}{r+n-a_r-1}}.$$

and  $\varphi_- > \varphi_+$ . Furthermore,  $|\varphi_+ - \varphi_-| \leq \frac{1}{n \binom{n-1}{r+n-a_r-1}}$  unless  $r+n-a_r-1 \in \{0, n-1\}$ . If  $r+n-a_r-1 = 0$  then  $r = 1$  and  $a_r = n$ , and so  $S = \{i, n\}$ . If  $r+n-a_r-1 = n-1$  then  $a_r = r$ , which is impossible.  $\square$

### C.6. Proof of Theorem 6.15

First, we recall the statement of Theorem 6.15.

**THEOREM 6.15.** *Let  $i$  be a positive integer. The function  $\varphi_i^{(\infty)}$  is continuous on  $(0, w(\infty))$ , and  $\lim_{q \rightarrow 0} \varphi_i^{(\infty)}(q) = \lim_{q \rightarrow w(\infty)} \varphi_i^{(\infty)}(q) = 0$ .*

**PROOF.** Let  $q \in (0, w(\infty))$ . We start by showing that  $\varphi_i^{(\infty)}$  is continuous from the right at  $q$ . Lemma 6.13 shows that we can find a subset  $P$  such that either  $q = w(P)$  or  $q \in (w(P^-), w(P)]$ . If  $q < w(P)$  then since  $\varphi_i^{(\infty)}$  is constant on  $(w(P^-), w(P)]$  according to Theorem 6.14, clearly  $\varphi_i^{(\infty)}$  is continuous from the right at  $q$ . Therefore we can assume that  $q = w(P)$ . Since  $q < w(\infty)$ , we can further assume that there are infinitely many  $n \notin P$ .

Suppose that we have a sequence  $q_j$  tending to  $q$  strictly from the right. For each  $j$  we can find a subset  $P_j$  such that either  $q_j = w(P_j)$  or  $q_j \in (w(P_j^-), w(P_j)]$ . We can assume that the second case doesn't happen by replacing  $q_j$  with  $w(P_j^-)$ ; the new sequence still tends to  $q$  strictly from the right. So we can assume that  $q_j = w(P_j) > w(P)$ . Let  $k(j) = \min(P_j \setminus P)$ , and let  $l(j) > k(j)$  be the smallest index larger than  $k(j)$  such that  $l(j) \notin P$ . Then

$$q_j - q = w(P_j) - w(P) \geq w_{k(j)} - \left( \sum_{t=k(j)+1}^{\infty} w_t - w_{l(j)} \right) \geq w_{l(j)}.$$

As  $j \rightarrow \infty$ ,  $l(j) \rightarrow \infty$  and so  $k(j) \rightarrow \infty$ . Therefore we can assume without loss of generality that  $k(j) > i$  for all  $j$ . Theorem 6.14 then implies that

$$|\varphi_i^{(\infty)}(q_j) - \varphi_i^{(\infty)}(q)| \leq \sum_{s=0}^{\infty} \frac{1}{(k(j) + s) \binom{k(j)+s-1}{s}} = \frac{1}{k(j) - 1},$$

using Lemma C.4. Since  $k(j) \rightarrow \infty$ ,  $\varphi_i^{(\infty)}(q_j) \rightarrow \varphi_i^{(\infty)}(q)$ .

We proceed to show that  $\varphi_i^{(\infty)}$  is continuous from the left at  $q$ . Lemma 6.13 shows that we can find a subset  $P$  such that either  $q = w(P)$  or  $q \in (w(P^-), w(P)]$ . In the second case, since  $\varphi_i^{(\infty)}$  is constant on  $(w(P^-), w(P)]$  according to Theorem 6.14, clearly  $\varphi_i^{(\infty)}$  is continuous from the left at  $q$ . Therefore we can assume that  $q = w(P)$ . Since  $q > 0$ , we can further assume that there are infinitely many  $n \in P$ .

Suppose that we have a sequence  $q_j$  tending to  $q$  strictly from the left. For each  $j$  we can find a subset  $P_j$  such that either  $q_j = w(P_j)$  or  $q_j \in (w(P_j^-), w(P_j)]$ , and in both cases  $q_j \leq w(P_j) < w(P)$ . Let  $k(j) = \min(P \setminus P_j)$ , and let  $l(j) > k(j)$  be the smallest index larger than  $k(j)$  such that  $l(j) \in P$ . Then

$$q - q_j \geq w(P) - w(P_j) \geq w_{k(j)} + w_{l(j)} - \sum_{t=k(j)+1}^{\infty} w_t \geq w_{l(j)}.$$

At this point we can prove that  $\varphi_i^{(\infty)}(q_j) \rightarrow \varphi_i^{(\infty)}(q)$  as in the preceding case.

It remains to show that  $\lim_{q \rightarrow 0} \varphi_i^{(\infty)}(q) = \lim_{q \rightarrow w(\infty)} \varphi_i^{(\infty)}(q) = 0$ . We start by showing that  $\lim_{q \rightarrow 0} \varphi_i^{(\infty)}(q) = 0$ . Let  $q_j$  be a sequence tending to 0 strictly from the right. As before, we can assume that  $q_j = w(P_j)$  for each  $j$ . Let  $k(j) = \min P_j$ . Since  $q_j \geq w_{k(j)}$ ,  $k(j) \rightarrow \infty$ . Therefore we can assume without loss of generality that  $k(j) > i$  for all  $j$ . Theorem 6.14 then implies that

$$\varphi_i^{(\infty)}(q_j) \leq \sum_{s=0}^{\infty} \frac{1}{(k(j) + s) \binom{k(j)+s-1}{s}} = \frac{1}{k(j) - 1},$$

using Lemma C.4. Since  $k(j) \rightarrow \infty$ ,  $\varphi_i^{(\infty)}(q_j) \rightarrow 0$ .

We finish the proof by showing that  $\lim_{q \rightarrow w(\infty)} \varphi_i^{(\infty)}(q) = 0$ . Let  $q_j$  be a sequence tending to  $M$  strictly from the left. As before, we can find subsets  $P_j$  such that  $q_j \leq w(P_j)$  and  $\varphi_i^{(\infty)}(q_j) = \varphi_i^{(\infty)}(w(P_j))$ . Let  $k(j)$  be the minimal  $k \notin P_j$ . Since  $q_j \leq w(\infty) - w_{k(j)}$ ,  $k(j) \rightarrow \infty$ . Therefore we can assume without loss of generality that  $k(j) > i$  for all  $j$ . Theorem 6.14 implies that

$$\varphi_i^{(\infty)}(q_j) \leq \frac{1}{i \binom{i-1}{i-1}} - \sum_{\ell=1}^{k(j)-1-i} \frac{1}{(i+\ell) \binom{i+\ell-1}{i+\ell-2}} = \frac{1}{k(j) - 1},$$

using Lemma C.4. Since  $k(j) \rightarrow \infty$ ,  $\varphi_i^{(\infty)}(q_j) \rightarrow 0$ .  $\square$