Shapley values in random weighted voting games

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Abstract

Shapley values, also known as Shapley–Shubik indices, measure the power that agents have in a weighted voting game. Suppose that agent weights are chosen randomly according to some distribution. We show that the expected Shapley values for the smallest and largest agent are independent of the quota for a large range of quotas, and converge exponentially fast to an explicit value depending only on the distribution. The proof makes a surprising use of renewal theory.

1 Introduction

Weighted voting games are specified by n agent weights $w_1, \ldots, w_n$ and a quota Q. A coalition of agents S is *winning* if $w(S) := \sum_{i \in S} w_i \geq Q$. Power indices measure the effect that each agent has on the decision-making process. The two most common ones are the Shapley–Shubik power index [Sha53b, SS54, SS69], which is a special case of the Shapley value [Sha53a], and the Banzhaf power index [Ban64]. See Felsenthal and Machover [FM98] for a thorough survey of this area.

Usually the weights are thought of as fixed. In this note we consider the case in which the weights are chosen at random. Jelnov and Tauman [TJ12] consider agent weights drawn from an exponential distribution. They show that the Shapley value of an agent is proportional to its weight in expectation.

We consider two main models:

1. The *natural iid* model: agent weights are drawn i.i.d. from some distribution X.

2. The *normalized iid* model: agent weights are drawn i.i.d. from some distribution X, and then normalized so that their sum is 1.

We show that for a wide range of quotas, the Shapley values of the smallest and largest agents in the natural iid model converge exponential fast to a value which depends only on the distribution:

Theorem 1.1. *Let X be a "reasonable" continuous random variable. For all n and for all $0 \ll Q \ll n \mathbb{E}[X]$, if we sample weights for an n-agent weighted voting game according to X then the Shapley values of the largest and smallest agents with respect to the quota Q have expected values*

$$\frac{1}{n} \mathop{\mathbb{E}}_{x \sim X_{\max}^n} \left[\frac{x}{\mathbb{E}[X_{\leq x}]} \right] + o(1), \qquad \frac{1}{n} \mathop{\mathbb{E}}_{x \sim X_{\min}^n} \left[\frac{x}{\mathbb{E}[X_{\geq x}]} \right] + o(1),$$

respectively, where the $o(1)$ terms are exponentially small, $X_{\max}^n$ ($X_{\min}^n$) is the distribution of a maximum (minimum) of n copies of X, and $X_{\leq x}$ ($X_{\geq x}$) is the distribution of X constrained on being at most x (at least x).

Here "reasonable" can be, for example, having finite support or exponential decay. Experimental results show that similar behavior is encountered in the normalized iid model.

Given a random variable X, define

$$\chi_{\min} = \sup\{x : \Pr[X < x] = 0\}, \quad \chi_{\max} = \inf\{x : \Pr[X > x] = 0\}.$$

Thus the "effective range" of X is $[\chi_{\min}, \chi_{\max}]$. Theorem 1.1 easily implies the following corollary:

Corollary 1.2. *Let X be a "reasonable" continuous random variable. For all $0 \ll q \ll \mathbb{E}[X]$, if we sample weights for an n-agent weighted voting game according to X then the Shapley values $\varphi_{\max}, \varphi_{\min}$ of the largest and smallest agents satisfy*

$$\lim_{n \to \infty} n\varphi_{\max}(qn) = \frac{\chi_{\max}}{\mathbb{E}[X]}, \quad \lim_{n \to \infty} n\varphi_{\min}(qn) = \frac{\chi_{\min}}{\mathbb{E}[X]}.$$

The proof of Theorem 1.1 makes surprising use of renewal theory.

Paper organization. The basic definitions are given in Section 2. The main theorem is sketched in Section 3. We work out two applications, to uniform and exponential distributions, in Section 4. We conclude with some conjectures in Section 5.

2 Definitions

We start with a formal definition of the Shapley and Banzhaf values in the case of weighted voting games:

Definition 2.1. A *weighted voting game* is given by the following data: number of agents n, agent weights $w_1, \ldots, w_n > 0$, and quota $0 \le q \le \sum_i w_i$. We sometimes think of agent weights as fixed and of the quota as varying.

A *coalition* of agents is a set of agents. For a coalition S we define $w(S) := \sum_{i \in S} w_i$. A coalition is *winning* if $w(S) \ge q$.

Let i be an agent. A *Shapley-random* coalition is obtained by generating a random permutation of the agents, and taking all agents preceding i in the permutation. A *Banzhaf-random* coalition is a uniformly random coalition not containing i.

The *Shapley value* of agent i at quota q, denoted $\varphi_i(q)$, is the probability that a Shapley-random coalition is not winning, but that adding i makes it winning. The *Banzhaf value* of agent i at quota q, denoted $\beta_i(q)$, is defined analogously.

There are simple formulas for both values:

Lemma 2.2. *Let $w_1, \ldots, w_n; q$ be a weighted voting game.*
The Shapley value of agent i at quota q is given by

$$\varphi_i(q) = \frac{1}{n} \sum_{j=1}^{n} \Pr_{\substack{i \notin S \\ |S|=j-1}} [q - w_i \le w(S) < q].$$

The Banzhaf value of agent i at quota q is given by

$$\beta_i(q) = \frac{1}{2^{n-1}} \sum_{j=1}^{n} \binom{n-1}{j-1} \Pr_{\substack{i \notin S \\ |S|=j-1}} [q - w_i \le w(S) < q].$$

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The formulas for the Shapley and Banzhaf values are very similar, but the different weights cause different behaviors. From now on we focus exclusively on the Shapley value.

Here are some pedestrian properties of the Shapley value:

Lemma 2.3. *Let $w_1, \ldots, w_n; q$ be a weighted voting game.*

(a) If $w_i \leq w_j$ then $\varphi_i(q) \leq \varphi_j(q)$.

(b) The Shapley values sum to 1.

In this paper our interest will focus on the extremal Shapley values:

Definition 2.4. *Let $w_1, \ldots, w_n; q$ be a weighted voting game, with minimum weight $w_{\min}$ and maximum weight $w_{\max}$. The smallest and largest Shapley values are given by*

$$\varphi_{\min}(q) = \min_i \varphi_i(q), \quad \varphi_{\max}(q) = \max_i \varphi_i(q),$$

Lemma 2.3 implies that $\varphi_{\min}$ ($\varphi_{\max}$) is the Shapley value corresponding to $w_{\min}$ ($w_{\max}$).

Finally, here are our two models for random weighted voting games:

Definition 2.5. *Let X be a random variable that is almost surely positive. Given a number n,*

(a) A random weighted voting game according to the natural iid model is obtained by drawing $w_i \sim X$.

(b) A random weighted voting game according to the normalized iid model is obtained by drawing $x_i \sim X$ and then taking $w_i = x_i / \sum_j x_j$, so that $\sum_i w_i = 1$.

We will use the following pieces of notation:

- $X_{\leq x}$ is the restriction of X to values at most x.

- $X_{\geq x}$ is the restriction of X to values at least x.

- $X_{\max}^n$ is the distribution of the maximum of n iid copies of X.

- $X_{\min}^n$ is the distribution of the minimum of n iid copies of X.

- $\chi_{\max}(X) = \inf\{x : \Pr[X > x] = 0\}$. If the set is empty then $\chi_{\max}(X) = \infty$.

- $\chi_{\min}(X) = \sup\{x : \Pr[X < x] = 0\}$.

3 Main theorem

Here is a more formal version of our main theorem:

Definition 3.1. *A random variable X is reasonable if:*

(a) X is continuous, not constant, and $X > 0$ almost surely.

(b) For some $C > 0$ and $\lambda < 1$, the density f of X satisfies $f(x) \leq C\lambda^x$.

(c) If $\chi_{\min}(X) = 0$, then $x = O(\mathbb{E}[X_{\leq x}])$ as $x \to 0$.

Theorem 3.2. *Let X be a reasonable random variable, and let $\epsilon > 0$. For all n and for all $\epsilon n \leq Q \leq (\mathbb{E}[X] - \epsilon)n$, a random weighted voting game according to the natural iid model satisfies*

$$\mathbb{E}[\varphi_{\max}(Q)] = \frac{1}{n} \mathop{\mathbb{E}}_{x \sim X_{\max}^n} \left[\frac{x}{\mathbb{E}[X_{\leq x}]} \right] + o(1), \quad \mathbb{E}[\varphi_{\min}(Q)] = \frac{1}{n} \mathop{\mathbb{E}}_{x \sim X_{\min}^n} \left[\frac{x}{\mathbb{E}[X_{\geq x}]} \right] + o(1).$$

Moreover, the $o(1)$ terms decay exponentially in n.

Proof sketch. We only prove the statement about $\varphi_{\max}$, the other statement being very similar.

Suppose without loss of generality that the largest agent is w_n, and take $x := w_n$. The marginal distribution of x is $X_{\max}^n$. Conditioned on the value of x, all other agent weights have distribution $X_{\leq x}$. Lemma 2.2 gives a formula for $\varphi_n(q)$ in terms of $w(S)$ for a random coalition of agents (excluding n). For each set S, the quantity $w(S)$ is distributed like the sum of $|S|$ copies of $X_{\leq x}$. Therefore

$$\mathbb{E}[n\varphi_{\max}(Q)] = \mathop{\mathbb{E}}_{x \sim X_{\max}^n} \left[\sum_{j=1}^{n} \mathop{\Pr}_{w_1,\ldots,w_{j-1} \sim X_{\leq x}} [Q - x \leq w_1 + \cdots + w_{j-1} < Q] \right]. \tag{1}$$

Since $Q \leq (1 - \epsilon)\mathbb{E}[X]$, for large n it is highly likely that the sum of n or more copies of $X_{\leq x}$ exceeds Q (unless x is very small, but that is improbable), and therefore

$$\mathbb{E}[n\varphi_{\max}(Q)] = \mathop{\mathbb{E}}_{x \sim X_{\max}^n} \left[\sum_{j=1}^{\infty} \mathop{\Pr}_{w_1,\ldots,w_{j-1} \sim X_{\leq x}} [Q - x \leq w_1 + \cdots + w_{j-1} < Q] \right] + o(1). \tag{2}$$

The renewal theorem shows that as $Q \to \infty$, the infinite sum converges to $x / \mathbb{E}[X_{\leq x}]$. Stone [Sto65] showed that the convergence is exponentially fast for exponentially decaying random variables. However, the error term in general depends on the distribution of holding times $X_{\leq x}$.

In order to evaluate (2) we thus need a uniform version of the renewal theorem. We will use the uniform renewal theorem of Blanchet and Glynn [BG07]. This theorem gives a uniform error term when the random variables are uniformly exponentially decaying, uniformly non-lattice, and their expectation is bounded away from zero. In our case these conditions hold for the class $\{X_{\leq x} : x \geq m\}$ for any $m > \chi_{\min}(X)$.

We make the arbitrary choice $m = \frac{\chi_{\min} + \chi_{\max}}{2}$. We split (2) into cases, depending on whether $x \geq m$ (the good case) or $x < m$ (the bad case). Since the bad case happens with very small probability, we obtain

$$\mathbb{E}[n\varphi_{\max}(Q)] = \mathop{\mathbb{E}}_{x \sim (X_{\max}^n)_{\geq m}} \left[\sum_{j=1}^{\infty} \mathop{\Pr}_{w_1,\ldots,w_{j-1} \sim X_{\leq x}} [Q - x \leq w_1 + \cdots + w_{j-1} < Q] \right] + o(1)$$

$$= \mathop{\mathbb{E}}_{x \sim (X_{\max}^n)_{\geq m}} \left[\frac{x}{\mathbb{E}[X_{\leq x}]} \right] + o(1). \tag{3}$$

If $\chi_{\min}(X) > 0$ (or in the case of $\varphi_{\min}$) then we can replace $(X_{\max}^n)_{\geq m}$ with $X_{\max}^n$, thus completing the proof. If $\chi_{\min}(X) = 0$ then it could potentially be that $x / \mathbb{E}[X_{\leq x}]$ blows up as $x \to 0$. However, this is ruled out by our assumption that X is reasonable, and so we obtain the stated formula. $\square$

We easily deduce the following corollary:

Corollary 3.3. *Let X be a reasonable random variable. For all $0 < q < \mathbb{E}[X]$, a random weighted voting game according to the natural iid model satisfies*

$$\lim_{n \to \infty} n\varphi_{\max}(qn) = \frac{\chi_{\max}(X)}{\mathbb{E}[X]}, \quad \lim_{n \to \infty} n\varphi_{\min}(qn) = \frac{\chi_{\min}(X)}{\mathbb{E}[X]}.$$

Proof. As $n \to \infty$, the distribution of $X_{\max}^n$ converges to $\chi_{\max}$, and that of $X_{\min}^n$ converges to $\chi_{\min}$. $\square$

4 Applications

4.1 Uniform distribution

Let $U(a, b)$ be the uniform distribution over the interval $[a, b]$, where $0 \leq a < b$. It is easy to verify that $U(a, b)$ is reasonable. Corollary 3.3 shows that under the natural iid model and for $0 < q < \frac{a+b}{2}$, $n\varphi_{\max}(qn) \to \frac{2b}{a+b}$ and $\varphi_{\min}(qn) \to \frac{2a}{a+b}$. This is illustrated in Figure 1 for the normalized iid model.

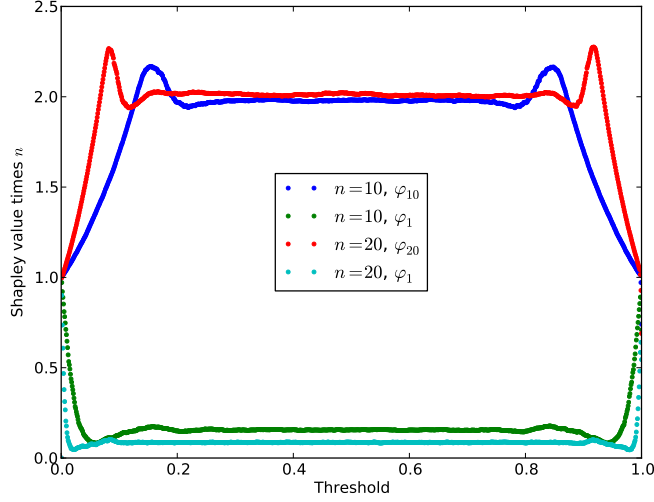The following theorem provides more refined estimates:

Figure 1: Shapley values for $X = U(0,1)$ and $n = 10, 20$ of both minimal and maximal agents, multiplied by n, for the normalized iid model. Results of 10^6 experiments.

Theorem 4.1. *Let $X = U(a, b)$, where $0 \leq a < b$. The following estimates hold, where the $o(1)$ terms decay exponentially fast:*

- *For all $0 < q < \frac{a+b}{2}$:*

$$\mathbb{E}[\varphi_{\max}(qn)] = \frac{1}{b-a} \int_a^b \left(\frac{t-a}{b-a}\right)^{n-1} \frac{2t}{a+t} \, dt + o(1).$$

We can evaluate the integral as a series:

$$\frac{1}{b-a} \int_a^b \left(\frac{t-a}{b-a}\right)^{n-1} \frac{2t}{a+t} \, dt = \frac{2b}{a+b} \frac{1}{n} - \frac{2a}{a+b} \sum_{d=1}^{\infty} \left(\frac{b-a}{a+b}\right)^d \frac{d!}{n(n+1)\cdots(n+d)}.$$

In particular, when $a = 0$ the integral is equal to $\frac{2}{n}$.

- *For all $0 < q < \frac{a+b}{2}$:*

$$\mathbb{E}[\varphi_{\min}(qn)] = \frac{1}{b-a} \int_a^b \left(\frac{b-t}{b-a}\right)^{n-1} \frac{2t}{b+t} \, dt + o(1).$$

We can evaluate the integral as a series:

$$\frac{1}{b-a} \int_a^b \left(\frac{b-t}{b-a}\right)^{n-1} \frac{2t}{b+t} \, dt = \frac{2a}{a+b} \frac{1}{n} - \frac{2b}{a+b} \sum_{d=1}^{\infty} \left(\frac{a-b}{a+b}\right)^d \frac{d!}{n(n+1)\cdots(n+d)}.$$

In particular, when $a = 0$ the integral is equal to

$$2\sum_{d=1}^{\infty} \frac{(-1)^{d+1}d!}{n(n+1)\cdots(n+d)} = \frac{2}{n(n+1)} - \frac{4}{n(n+1)(n+2)} + \cdots.$$

Proof. We have $\chi_{\min} = a$, $\chi_{\max} = b$, $\mathbb{E}[X] = (a+b)/2$, $\mathbb{E}[X_{\leq x}] = (a+x)/2$ and $\mathbb{E}[X_{\geq x}] = (b+x)/2$.
The distribution of $X_{\max}^n$ is given by

$$\Pr[X_{\max}^n \leq t] = \Pr[X \leq t]^n = \left(\frac{t-a}{b-a}\right)^n.$$

The corresponding density function is the derivative $\frac{n}{b-a}\left(\frac{t-a}{b-a}\right)^{n-1}$. The formula for $\varphi_{\max}(qn\,\mathbb{E}[X])$ follows from

$$\underset{x \sim X_{\max}^n}{\mathbb{E}}\left[\frac{x}{\mathbb{E}[X_{\leq x}]}\right] = \underset{x \sim X_{\max}^n}{\mathbb{E}}\left[\frac{2x}{a+x}\right] = \frac{1}{b-a}\int_a^b n\left(\frac{t-a}{b-a}\right)^{n-1}\frac{2t}{a+t}\,dt.$$

Similarly, the distribution of $X_{\min}^n$ is given by

$$\Pr[X_{\min}^n \geq t] = \Pr[X \geq t]^n = \left(\frac{b-t}{b-a}\right)^n.$$

The corresponding density function is the negated derivative $\frac{n}{b-a}\left(\frac{b-t}{b-a}\right)^{n-1}$. The formula for $\varphi_{\min}(qn\,\mathbb{E}[X])$ follows from

$$\underset{x \sim X_{\min}^n}{\mathbb{E}}\left[\frac{x}{\mathbb{E}[X_{\geq x}]}\right] = \underset{x \sim X_{\min}^n}{\mathbb{E}}\left[\frac{2x}{b+x}\right] = \frac{1}{b-a}\int_a^b n\left(\frac{b-t}{b-a}\right)^{n-1}\frac{2t}{b+t}\,dt.$$

We proceed to evaluate the integrals, starting with the first one. The basic observation is

$$\frac{1}{b-a}\int_a^b \left(\frac{t-a}{b-a}\right)^{n-1}\left(\frac{b-t}{b-a}\right)^d dt = \int_0^1 s^{n-1}(1-s)^d\,ds = \frac{d!}{n(n+1)\cdots(n+d)}, \tag{4}$$

using the substitution $s = (t-a)/(b-a)$ and the Beta integral. A simple calculation shows that

$$\frac{t}{a+t} = \frac{b}{a+b} - \frac{a}{a+b}\sum_{d=1}^{\infty}\left(\frac{b-t}{a+b}\right)^d.$$

Therefore, using (4),

$$\frac{1}{b-a}\int_a^b \left(\frac{t-a}{b-a}\right)^{n-1}\frac{2t}{a+t}\,dt$$

$$= \frac{2}{b-a}\int_a^b \left(\frac{t-a}{b-a}\right)^{n-1}\left[\frac{b}{a+b} - \frac{a}{a+b}\sum_{d=1}^{\infty}\left(\frac{b-t}{a+b}\right)^d\right]dt$$

$$= \frac{2b}{a+b}\frac{1}{n} - \frac{2a}{a+b}\sum_{d=1}^{\infty}\left(\frac{b-a}{a+b}\right)^d\frac{d!}{n(n+1)\cdots(n+d)}.$$

The second integral can be evaluated in the same way. Alternatively, substitute $(a,b) = (b,a)$ in the formula for the first integral to obtain

$$\frac{1}{b-a}\int_a^b \left(\frac{b-t}{b-a}\right)^{n-1}\frac{2t}{b+t}\,dt = \frac{1}{a-b}\int_b^a \left(\frac{t-b}{a-b}\right)^{n-1}\frac{2t}{b+t}\,dt$$

$$= \frac{2a}{a+b}\frac{1}{n} - \frac{2b}{a+b}\sum_{d=1}^{\infty}\left(\frac{a-b}{a+b}\right)^d\frac{d!}{n(n+1)\cdots(n+d)}. \qquad \square$$
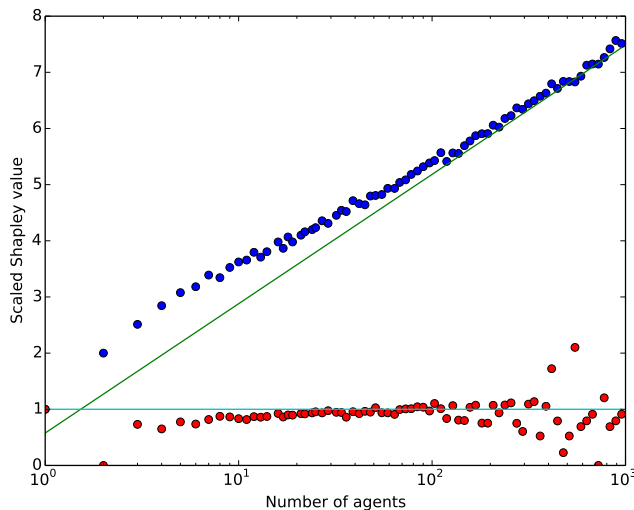
Figure 2: Shapley values for the normalized iid model with $X = \text{Exp}(1)$ and various n of both maximal and minimal agent, multiplied by n and by n^2, respectively, at the quota $q = 1/2$. Results of 10^6 experiments. The experimental results are compared to the predictions of Theorem 4.2: the maximal Shapley value is compared against $(\log n + \gamma)/n$, and the minimal Shapley value is compared against $1/n^2$.

4.2 The exponential distribution

Let $\text{Exp}(1)$ be the standard exponential distribution with expectation 1. It is easy to verify that $\text{Exp}(1)$ is reasonable; the last condition in the definition holds since

$$\mathbb{E}[X_{\leq x}] = \frac{\int_0^x e^{-t}t\,dt}{\int_0^x e^{-t}\,dt} = \frac{1 - (x+1)e^{-x}}{1 - e^{-x}} = \frac{(3/2)x^2 + O(x^3)}{x + O(x^2)} = \frac{3}{2}x + O(x^2).$$

Corollary 3.3 shows that under the natural iid model and for positive q, $n\varphi_{\max}(qn) \to \infty$ and $n\varphi_{\min}(qn) \to 0$.

The following theorem provides more useful estimates, which are illustrated by Figure 2:

Theorem 4.2. *Let $X = \text{Exp}(1)$. The following estimates hold, where the $o(1)$ terms decay exponentially fast:*

- *For all $0 < q < 1$,*

$$\mathbb{E}[\varphi_{\max}(qn)] = \int_0^\infty (1 - e^{-x})^n \frac{x}{e^x - (1+x)}\,dx + o(1),$$

and the integral satisfies

$$\int_0^\infty (1 - e^{-x})^n \frac{x}{e^x - (1+x)}\,dx = \frac{\log n + \gamma}{n} + O\left(\frac{\log^2 n}{n^2}\right).$$

- *For all $0 < q < 1$,*

$$\mathbb{E}[\varphi_{\min}(qn)] = \int_0^\infty e^{-nx} \frac{x}{x+1}\,dx + o(1),$$

and the integral satisfies

$$\int_0^\infty e^{-nx} \frac{x}{x+1}\,dx = \frac{1}{n^2} - O\left(\frac{1}{n^3}\right).$$

Proof. Notice first that
$$\mathbb{E}[X_{\leq x}] = \frac{\int_0^x e^{-t} t\, dt}{\int_0^x e^{-t}\, dt} = \frac{1 - (x+1)e^{-x}}{1 - e^{-x}}.$$

This formula shows that
$$\phi(x) := \frac{x}{\mathbb{E}[X_{\leq x}]} = \frac{x(1 - e^{-x})}{1 - (x+1)e^{-x}}.$$

It is easy to calculate $\Pr[X_{\max}^n \leq x] = (1 - e^{-x})^n$, and so the density of $X_{\max}^n$ is $n(1 - e^{-x})^{n-1}e^{-x}$. We conclude that
$$\mathop{\mathbb{E}}_{x \sim X_{\max}^n}[\phi(x)] = n \int_0^\infty (1 - e^{-x})^n \frac{x}{e^x - (1+x)}\, dx$$
$$= n \int_0^1 (1 - t)^n \log \frac{1}{t} \frac{dt}{1 - (t + t \log \frac{1}{t})}.$$

In order to estimate the integral, write
$$\int_0^1 (1 - t)^n \log \frac{1}{t} \frac{dt}{1 - (t + t \log \frac{1}{t})} = I_n + J_n + K_n,$$

where I_n, J_n, K_n are given by
$$I_n = \int_0^1 (1 - t)^n \log \frac{1}{t}\, dt,$$
$$J_n = \int_0^1 (1 - t)^n \log \frac{1}{t}(t + t \log \frac{1}{t})\, dt,$$
$$K_n = \int_0^1 (1 - t)^n \log \frac{1}{t}(t + t \log \frac{1}{t})^2 \frac{dt}{1 - (t + t \log \frac{1}{t})}.$$

Surprisingly, we can calculate I_n, J_n exactly in terms of the harmonic numbers H_n:
$$I_n = \frac{H_{n+1}}{n+1} = \frac{1}{n+1} \sum_{i=0}^n \frac{1}{i+1}, \tag{5}$$
$$J_n = \frac{1}{(n+1)(n+2)} \sum_{i=0}^n \frac{2}{i+2} H_i - \frac{i^2 - i - 4}{(i+1)(i+2)^2}. \tag{6}$$

In order to get the formula for I_n, notice first that $t + t \log \frac{1}{t}$ is an antiderivative of $\log \frac{1}{t}$. This immediately implies that $I_0 = 1$, and for $n > 0$, integration by parts gives
$$I_n = \int_0^1 (1 - t)^n \log \frac{1}{t}\, dt$$
$$= (1 - t)^n \left(t + t \log \frac{1}{t}\right)\Big|_0^1 + n \int_0^1 (1 - t)^{n-1} t \left(1 + \log \frac{1}{t}\right) dt$$
$$= n \int_0^1 [(1 - t)^{n-1} - (1 - t)^n] \log \frac{1}{t}\, dt + n \int_0^1 (1 - t)^{n-1} t\, dt$$
$$= n(I_{n-1} - I_n) + \frac{1}{n+1},$$

using the formula for the Beta integral. This shows that $(n+1)I_n = nI_{n-1} + 1/(n+1)$, which implies formula (5). Formula (6) is proved along the same lines. It is well-known that $H_n = \log n + \gamma + O(1/n)$, and this shows that
$$I_n = \frac{H_{n+1}}{n+1} = \frac{\log(n+1) + \gamma + O(1/n)}{n+1} = \frac{\log n + \gamma}{n} + O\left(\frac{\log n}{n^2}\right).$$

Similarly, using the integral $\int_1^n \frac{2\log m}{m}\,dm = \log^2 n$ to estimate the corresponding series, we obtain

$$J_n = \frac{1}{\Theta(n^2)}\sum_{i=0}^{n} O\left(\frac{\log n}{n}\right) = O\left(\frac{\log^2 n}{n^2}\right).$$

It remains to estimate K_n. We break K_n into two parts, $L_n = \int_0^{1/e}$ and $M_n = \int_{1/e}^{1}$, which we bound separately. Since $t + t\log\frac{1}{t}$ is increasing, when $t \le 1/e$ we have $t + t\log\frac{1}{t} \le 2/e < 1$, and so

$$L_n = \int_0^{1/e}(1-t)^n \log\frac{1}{t}(t + t\log\tfrac{1}{t})^2\frac{dt}{1-(t+t\log\frac{1}{t})} \le \frac{1}{1-2/e}J_n = O\left(\frac{\log^2 n}{n^2}\right).$$

When $t \ge 1/e$, we have $\log\frac{1}{t} \le 1$ and so

$$M_n \le 4\int_{1/e}^{1}(1-t)^n\frac{dt}{1-(t+t\log\frac{1}{t})} = 4\int_0^{1-1/e} s^n\frac{ds}{s+(1-s)\log(1-s)},$$

where we applied the substitution $s = 1 - t$. Taylor expansion shows that $s + (1-s)\log(1-s) \ge s^2/2$, and so

$$M_n \le 8\int_0^{1-1/e} s^{n-2}\,ds = 8\frac{(1-1/e)^{n-1}}{n-1}.$$

We conclude that $J_n + K_n = O(\log^2 n/n^2)$, and so

$$\int_0^1 (1-t)^n \log\frac{1}{t}\frac{dt}{1-(t+t\log\frac{1}{t})} = \frac{\log n + \gamma}{n} + O\left(\frac{\log^2 n}{n^2}\right).$$

We move on to calculate $\mathbb{E}[\varphi_{\min}(qn\,\mathbb{E}[X])]$. We have

$$\mathbb{E}[X_{\ge x}] = \frac{\int_x^\infty e^{-t}t\,dt}{\int_x^\infty e^{-t}\,dt} = \frac{(x+1)e^{-x}}{e^{-x}} = x + 1.$$

It is easy to calculate $\Pr[X_{\min}^n \ge x] = e^{-nx}$, and so the density of $X_{\min}^n$ is ne^{-nx}. We conclude that

$$\mathbb{E}_{x\sim X_{\min}^n}\left[\frac{x}{\mathbb{E}[X_{\ge x}]}\right] = n\int_0^\infty e^{-nx}\frac{x}{x+1}\,dx.$$

This gives us the stated formula. In order to estimate the integral, note that

$$\int_0^\infty e^{-nx}\frac{x}{x+1}\,dx = \int_0^\infty e^{-nx}\,dx - \int_0^\infty e^{-nx}\frac{dx}{x+1}$$
$$= \frac{1}{n} - e^n\int_1^\infty e^{-nx}\frac{dx}{x} = \frac{1}{n} - e^n\int_n^\infty \frac{e^{-x}}{x}\,dx$$

The latter integral is an exponential integral, and its asymptotic expansion is

$$\int_n^\infty \frac{e^{-x}}{x}\,dx = e^{-n}\left(\frac{1}{n} - \frac{1}{n^2} + O\left(\frac{1}{n^3}\right)\right).$$

We conclude that

$$\int_0^\infty e^{-nx}\frac{x}{x+1}\,dx = \frac{1}{n^2} - O\left(\frac{1}{n^3}\right). \qquad \square$$

5 Conjectures

5.1 Normalized iid model

Theorem 3.2 is stated for the natural iid model. However, our experimental results (appearing in Figure 1 and Figure 2) suggest that it holds for the normalized iid model as well:

Conjecture 5.1. *Let X be a reasonable random variable, and let $\epsilon > 0$. For all n and for all $\epsilon \leq q \leq 1 - \epsilon$, a random weighted voting game according to the normalized iid model satisfies*

$$\mathbb{E}[\varphi_{\max}(q)] = \frac{1}{n} \mathop{\mathbb{E}}_{x \sim X_{\max}^n} \left[\frac{x}{\mathbb{E}[X_{\leq x}]} \right] + o(1), \qquad \mathbb{E}[\varphi_{\min}(q)] = \frac{1}{n} \mathop{\mathbb{E}}_{x \sim X_{\min}^n} \left[\frac{x}{\mathbb{E}[X_{\geq x}]} \right] + o(1).$$

Moreover, the $o(1)$ terms decay exponentially in n.

The conjecture would imply the following corollary:

Conjecture 5.2. *Let X be a reasonable continuous random variable. For all $0 < q < 1$, a random weighted voting game according to the normalized iid model satisfies*

$$\lim_{n \to \infty} n\varphi_{\max}(q) = \frac{\chi_{\max}(X)}{\mathbb{E}[X]}, \quad \lim_{n \to \infty} n\varphi_{\min}(q) = \frac{\chi_{\min}(X)}{\mathbb{E}[X]}.$$

The difficulty in proving Conjecture 5.1 lies in the dependence between $Q := q \sum_i w_i$ and the weights $w_1, \ldots, w_{n-1}$.

5.2 Intermediate Shapley values

Theorem 3.2 and Corollary 3.3 describe the behavior of the extreme Shapley values. It is natural to ask how the Shapley values in between behave. Based on the proof of Theorem 3.2, we can formulate a conjectured answer to this question:

Conjecture 5.3. *Let X be a reasonable random variable, and let $\epsilon > 0$. For all n, for all $0 < p < 1$, and for all $\epsilon \leq q \leq 1 - \epsilon$, a random weighted voting game according to the normalized iid model satisfies*

$$\mathbb{E}[\varphi'_{pn}(q)] = \frac{1}{n} \mathop{\mathbb{E}}_{x \sim X_{pn}^n} \left[\frac{x}{\mathbb{E}[X_{p,x}]} \right] + o(1),$$

where $\varphi'_{pn}(q)$ is the Shapley value of the pn'th smallest agent at the quota q, X_{pn}^n is the distribution of the pn'th order statistic of n iid copies of X, and $X_{p,x}$ is a mixture of $X_{\leq x}$ (with probability p) and of $X_{\geq x}$ (with probability $1 - p$).

Moreover, the $o(1)$ terms decay exponentially in n.

The conjecture would imply the following corollary:

Conjecture 5.4. *Let X be a reasonable random variable. For all $0 < p, q < 1$, a random weighted game according to the normalized iid model satisfies*

$$\lim_{n \to \infty} n\varphi'_{pn}(q) = \frac{x}{\mathbb{E}[X]}, \quad \text{where } \Pr[X \leq x] = p.$$

This conjecture is corroborated by the experimental evidence displayed in Figure 3.
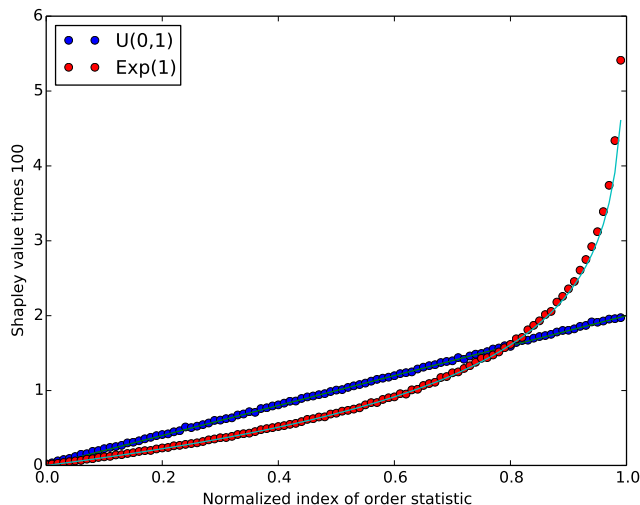
Figure 3: All Shapley values for $X = U(0,1)$ and $X = \text{Exp}(1)$ (both in the normalized iid model) and the setting $n = 100$ at the quota $q = 1/2$, normalized by n. Results of 10^6 experiments. The experimental results are compared to the predictions of Conjecture 5.4: $\varphi_{pn}(q) = 2p$ for $X = U(0,1)$ and $\varphi_{pn}(q) = -\log(1-p)$ for $X = \text{Exp}(1)$.

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