Asymptotic performance of the Grimmett–McDiarmid heuristic

Yuval Filmus

November 11, 2020

Abstract

Grimmett and McDiarmid analyzed a simple heuristic for finding stable sets in random graphs (suggested earlier by Johnson). They showed that the heuristic finds a stable set of size roughly $\log_2 n$ probability, on a $G(n, 1/2)$ random graph, with high probability. We determine the asymptotic distribution of the size of the stable set found by the algorithm.

1 Introduction

Grimmett and McDiarmid [GM75] considered the problem of coloring $G(n, 1/2)$ random graphs. As part of their solution, they suggested the following simple greedy heuristic for finding a large stable set: scan the vertices in random order, adding to the stable set any vertex which is not adjacent to the vertices added so far. They showed that this heuristic algorithm constructs a stable set of size roughly $\log_2 n$, with high probability. In contrast, the maximum stable set in the graph has size roughly $2\log_2 n$, with high probability, and is concentrated on one or two values [Mat72, BE76, Mat76]. (This contrasts with the non-concentration of the chromatic number, shown recently by Heckel [Hec20].)

Karp [Kar76] concluded that the Grimmett–McDiarmid algorithm (which had been suggested independently by Johnson [Joh74]) gives a 2-approximation to the maximum stable set problem in $G(n, 1/2)$ random graphs, with high probability. He asked whether this approximation ratio can be improved to $2 - \epsilon$ for any $\epsilon > 0$. Despite some lower order improvements [KS98], the problem remains open. (The planted clique problem [Jer92, Kuč95], an attempt to mitigate this difficulty, is beyond the scope of this work.)

Grimmett and McDiarmid showed that for every $\epsilon > 0$, with high probability their algorithm constructs a stable set whose size is between $(1 - \epsilon)\log_2 n$ and $(1 + \epsilon)\log_2 n$. Their bounds were later improved [McD79, McD84, BT85] in the context of analyzing algorithms for coloring random graphs. However, to the best of our knowledge, an analysis of the limiting distribution of the size has never been published.[1] This is our goal in this work.

Let us briefly indicate how to analyze the Grimmett–McDiarmid algorithm. Denote by N_k the number of remaining vertices not adjacent to the first k vertices in the stable set constructed by the algorithm, or zero if the algorithm terminated before choosing k vertices. A simple induction shows that $\mathbb{E}[N_k] \leq n/2^k$, and so with high probability, the algorithm produces a stable set of size at most $\log_2 n + f(n)$, where $f(n)$ is *any* function satisfying $f(n) \to \infty$.

For the lower bound, let us imagine that there are infinitely many vertices (this idea already appears in [GM75]), let $i_0 = 0$, and let i_k be the index of the k'th chosen vertex in the random order of the vertices (starting with 1). Then $i_{k+1} - i_k \sim \mathrm{G}(2^{-k})$ (geometric random variable with success probability 2^{-k}), and the size of the clique is the maximal k such that $i_k \leq n$. It is easy to calculate $\mathbb{E}[i_k] = 2^k - 1$, from which it easily follows that with high probability, the algorithm produces a stable set of size at least $\log_2 n - f(n)$, where $f(n)$ is *any* function satisfying $f(n) \to \infty$.

Let $\mathbf{k}$ be the size of the stable set produced by the algorithm. The foregoing suggests that $\mathbf{k} - \log_2 n$ approaches a limiting distribution, but there is a complication: $\mathbf{k}$ is always an integer, while the fractional part of $\log_2 n$ varies. We will show that if we fix the fractional part $\{\log_2 n\}$ then $\mathbf{k} - \log_2 n$ indeed approaches a limit; and furthermore, the various limits stem from the same continuous distribution.

[1] In unpublished work, Huang [Hwa08] worked out the asymptotic moment generating function of the deviation of the size from $\log_2 n$.

Definition 1.1. The random variable $\mathbf{H}$ is given by the following sum of exponential distributions:

$$\mathbf{H} = \sum_{i=1}^{\infty} \mathrm{E}(2^i),$$

where $\mathrm{E}(2^i)$ is an exponential random variable with mean 2^{-i}. (This defines a random variable due to Kolmogorov's three-series theorem [Fel71, VIII.5,IX.9].)

Theorem 1.2. *For a given n, define*

$$p_k = \Pr[\mathbf{k} = k], \quad q_k = \Pr\left[\frac{n}{2^{k+1}} \leq \mathbf{H} < \frac{n}{2^k}\right].$$

Then we have

$$\sum_{k=0}^{\infty} |p_k - q_k| = o(1).$$

Prodinger [Pro92, Pro93a] mentions that the distribution of $\mathbf{k}$ is identical to the distribution of the Morris approximate counter [Mor78], thoroughly analyzed by Flajolet [Fla85]. In particular, Theorem 1.2 is very similar to [Fla85, Proposition 3].

The existence of a limiting distribution in the sense of Theorem 1.2 also follows from the work of Janson, Lavault, and Louchard [JLL08] on leader election algorithms; see also [Pro93a, Pro93b, FMS96, Kne01, LP08].

Background on stable set algorithms The first heuristic algorithms for finding stable sets appear in early work from the 1960s on scheduling [Col64, WP67], as an ingredient of graph coloring algorithms. These heuristics ("non-adaptive degree-greedy") scan the vertices in increasing order of degree, adding each vertex not adjacent to vertices added so far. Matula [Mat68] and Kučera [Kuč77] suggest an adaptive version of this heuristic ("adaptive degree-greedy"), which repeatedly adds a feasible vertex of minimal degree. These heuristics and others were evaluated empirically on random graphs in [MMI72, BT85]. Kučera [Kuč92] analyzed some of these heuristics with a cryptographic application in mind.

The work of Grimmett and McDiarmid [GM75] was the first to analyze any heuristic for stable set or coloring. While aware of more sophisticated heuristics, they were only able to analyze the "random-greedy" heuristic which is the focus of this work, suggested independently by Johnson [Joh74]. McDiarmid [McD84] showed that the adaptive degree-greedy heuristic also produces stable sets of size at least $\log_2 n$, but was unable to improve on that due to "awkward conditioning problems". To the best of our knowledge, the suspicion that the adaptive degree-greedy heuristic improves on the random-greedy heuristic remains unproven.

Other heuristics appear in the literature. For example, Matula et al. [MMI72] and Brockington and Culberson [BC96] suggested further degree-greedy heuristics, Jerrum [Jer92] suggested the Metropolis algorithm, and Krivelevich and Vu [KV02] (see also [COT04]) considered running the greedy coloring algorithm and taking the largest color class. So far the only algorithm which provably improves on the random-greedy heuristic is due to Krivelevich and Sudakov [KS98], which runs Grimmett–McDiarmid on half the vertices, and then switches to exhaustive search. This algorithm results in a stable set of size $\log_2 n + \Theta(\sqrt{\log n})$. Consult [BBPP99] for a survey of many heuristics.

Further information on the stable set problem and the related graph coloring problem can be found in the surveys by Frieze and McDiarmid [FM97], Krivelevich [Kri02], and Kang and McDiarmid [KM15], as well as in standard textbooks on random graph theory [Pal88, JŁR00, Bol01, FK16].

Acknowledgements The author is a Taub fellow, and is supported by the Taub Foundation and ISF grant 1337/16. The author would like to thank the reviewers for their helpful suggestions, and Hsien-Kuei Hwang for mentioning the relevance of leader election algorithms.

Preliminaries The Wasserstein distance $W_1(X, Y)$ between two random variables is the minimum of $\mathbb{E}[|X-Y|]$ over all couplings of X, Y. This formula shows that W_1 is subadditive: $W_1(X_1+X_2, Y_1+Y_2) \leq W_1(X_1, Y_1)+W_1(X_2, Y_2)$. The Wasserstein distance is also given by the explicit formula [PZ20, Cor. 1.5.3]

$$W_1(X, Y) = \int_{-\infty}^{\infty} |\Pr[X < t] - \Pr[Y < t]|\, dt.$$

The Kolmogorov–Smirnov distance between X and Y is $\sup_t |\Pr[X < t] - |\Pr[Y < t]|$.

Lemma 1.3 ([CR07, Lemma 2])**.** *If Y is a continuous random variable with density bounded by C, then the Kolmogorov–Smirnov distance between X and Y is bounded by $2\sqrt{CW_1(X,Y)}$.*

Proof. We will show that $|\Pr[X < t] - \Pr[Y < t]|$ holds for every t. Fix an arbitrary point t, and let $\epsilon > 0$ be a parameter to be chosen. Define a real-valued function f as follows: $f(x) = 1$ for $x \le t$, $f(x) = 1 - (x-t)/\epsilon$ for $t \le x \le t+\epsilon$, and $f(x) = 0$ for $x \ge t+\epsilon$. Clearly $\Pr[X < t] \le \mathbb{E}[f(X)]$. Since Y has density bounded by C, we have $\Pr[Y < t] \ge \Pr[Y < t+\epsilon] - C\epsilon \ge \mathbb{E}[f(Y)] - C\epsilon$. Thus

$$\Pr[X < t] - \Pr[Y < t] \le \mathbb{E}[f(X) - f(Y)] + C\epsilon.$$

On the other hand, since f is $1/\epsilon$-Lipschitz, clearly $\mathbb{E}[f(X) - f(Y)] \le W_1(X,Y)/\epsilon$, and so

$$\Pr[X < t] - \Pr[Y < t] \le W_1(X,Y)/\epsilon + C\epsilon.$$

Choosing $\epsilon = \sqrt{W_1(X,Y)/C}$, we get the required upper bound on $\Pr[X < t] - \Pr[Y < t]$.

The lower bound is proved in a similar way. Define a real-valued function g as follows: $g(x) = 1$ for $x \le t-\epsilon$, $g(x) = (x-(t-\epsilon))/\epsilon$ for $t-\epsilon \le x \le t$, and $g(x) = 0$ for $x \ge t$. This time $\Pr[X < t] \ge \mathbb{E}[g(X)]$ while $\Pr[Y < t] \le \Pr[Y < t-\epsilon] + C\epsilon \le \mathbb{E}[g(Y)] + C\epsilon$. Thus

$$\Pr[Y < t] - \Pr[X < t] \le \mathbb{E}[g(Y) - g(X)] + C\epsilon \le W_1(X,Y)/\epsilon + C\epsilon \le 2\sqrt{CW_1(X,Y)}. \qquad \square$$

2 Proof

Recall that $\mathbf{k}$ is the size of the stable set produced by the Grimmett–McDiarmid algorithm. Grimmett and McDiarmid proved the following result, whose proof was outlined in the introduction.

Lemma 2.1.

$$\Pr[\mathbf{k} < k] = \Pr[\mathrm{G}(1) + \mathrm{G}(1/2) + \cdots + \mathrm{G}(1/2^{k-1}) > n] = \Pr[\mathrm{G}(1/2) + \cdots + \mathrm{G}(1/2^{k-1}) \ge n].$$

Our main idea is to rewrite this formula as follows:

$$\Pr[\mathbf{k} < k] = \Pr\left[\frac{\mathrm{G}(1/2^{k-1})}{n} + \frac{\mathrm{G}(1/2^{k-2})}{n} + \cdots + \frac{\mathrm{G}(1/2)}{n} \ge 1\right]. \tag{1}$$

It is known that the distribution $\mathrm{G}(c/n)/n$ tends (in an appropriate sense) to an exponential random variable $\mathrm{E}(c)$ [Fel71, Problem XIII.1]. We will show this quantitatively, in terms of the Wasserstein metric W_1.

Lemma 2.2. *If $p \le 1/2$ then*

$$W_1(\mathrm{G}(p)/n, \mathrm{E}(pn)) \le \frac{2}{n}.$$

Proof. Let $X = \lceil \mathrm{E}(pn)n \rceil$. Then for integer t,

$$\Pr[X \ge t] = \Pr[\mathrm{E}(pn) > (t-1)/n] = e^{-p(t-1)}.$$

In contrast,

$$\Pr[\mathrm{G}(p) \ge t] = (1-p)^{t-1}.$$

By construction, $W_1(X/n, \mathrm{E}(pn)) \le 1/n$, and so subadditivity of W_1 and the explicit formula for W_1 show that

$$W_1(\mathrm{G}(p)/n, \mathrm{E}(pn)) \le \frac{1}{n} + W_1(\mathrm{G}(p)/n, X/n) = \frac{1}{n} + \int_0^\infty \left|\Pr[\mathrm{G}(p)/n \ge s] - \Pr[X/n \ge s]\right| ds =$$
$$\frac{1}{n} + \frac{1}{n}\sum_{r=1}^{\infty} \left|\Pr[\mathrm{G}(p) \ge r] - \Pr[X \ge r]\right| = \frac{1}{n} + \frac{1}{n}\sum_{t=1}^{\infty} |(1-p)^t - e^{-pt}|.$$

Since $p \le 1/2$, we have $-p - p^2 \le \log(1-p) \le -p$, and so

$$e^{-pt-p^2t} \le (1-p)^t \le e^{-pt}.$$

Therefore, $e^{-x} \ge 1 - x$ implies that

$$|(1-p)^t - e^{-pt}| = e^{-pt} - (1-p)^t \le e^{-pt}(1 - e^{-p^2t}) \le p^2 t e^{-pt}.$$

We can thus bound

$$\sum_{t=1}^{\infty} |(1-p)^t - e^{-pt}| \le p^2 \sum_{t=1}^{\infty} \frac{t}{e^{pt}} = \frac{p^2 e^p}{(e^p - 1)^2} \le 1,$$

where the last step follows from

$$pe^{p/2} = \sum_{k=1}^{\infty} \frac{p^k}{2^{k-1}(k-1)!} \le \sum_{k=1}^{\infty} \frac{p^k}{k!} = e^p - 1,$$

which implies that $p^2 e^p \le (e^p - 1)^2$ for all $p \ge 0$. $\square$

Since W_1 is subadditive, we immediately conclude the following:

Lemma 2.3. *Let* $\mathbf{G} = \frac{1}{n} \cdot \big(\mathrm{G}(1/2^{k-1}) + \mathrm{G}(1/2^{k-2}) + \cdots + \mathrm{G}(1/2)\big)$. *Then for every* $k \ge 1$,

$$W_1\left(\frac{n}{2^k}\mathbf{G}, \mathbf{H}\right) \le \frac{k}{2^{k-1}}.$$

When $k = 0$ *(and so* $\mathbf{G}$ *is identically zero), this holds with the bound* 1.

Proof. Lemma 2.2 and subadditivity of W_1 show that

$$W_1(\mathbf{G}, \mathrm{E}(n/2^{k-1}) + \cdots + \mathrm{E}(n/2)) \le \frac{2(k-1)}{n},$$

which implies that

$$W_1\left(\frac{n}{2^k}\mathbf{G}, \mathrm{E}(2) + \cdots + \mathrm{E}(2^{k-1})\right) \le \frac{k-1}{2^{k-1}}.$$

On the other hand,

$$W_1\left(\sum_{\ell=k}^{\infty} \mathrm{E}(2^\ell), \mathbf{0}\right) = \mathbb{E}\left[\sum_{\ell=k}^{\infty} \mathrm{E}(2^\ell)\right] = \frac{1}{2^{k-1}},$$

where $\mathbf{0}$ is the constant zero random variable. The lemma follows for $k \ge 1$ from another application of subadditivity of W_1. When $k = 0$, the final step shows that $W_1(\mathbf{H}, \mathbf{0}) = 1$. $\square$

In order to convert this bound to a bound on the Kolmogorov–Smirnov distance using Lemma 1.3, we need to know that $\mathbf{H}$ is continuous and has a bounded density function.

Lemma 2.4. *The random variable* $\mathbf{H}$ *is continuous, and has a bounded density function* f*:*

$$f(x) = 2C^{-1} \sum_{i=1}^{\infty} (-1)^{i-1} e^{-2^i x} \prod_{r=1}^{i-1} \frac{2}{2^r - 1}, \text{ where } C = \prod_{s=1}^{\infty} (1 - 2^{-s}) > 0.$$

(The constant C *is the limit of the probability that an* $n \times n$ *matrix over* $GF(2)$ *is regular.)*

Proof. Let $\mathbf{H}^{(\ell)} = \sum_{i=1}^{\ell} \mathrm{E}(2^i)$. It is well-known [Fel71, Problem I.12] that the density of $\mathbf{H}^{(\ell)}$ is

$$f_\ell(x) = \sum_{i=1}^{\ell} 2^i e^{-2^i x} K_{\ell,i}, \text{ where } K_{\ell,i} = \prod_{\substack{j=1 \\ j \ne i}}^{\ell} \frac{2^j}{2^j - 2^i}.$$

Note that

$$K_{\ell,i} = (-1)^{i-1} \prod_{j=1}^{i-1} \frac{1}{2^{i-j}-1} \times \prod_{j=i+1}^{\ell} \frac{1}{1-2^{i-j}} = (-1)^{i-1} \prod_{r=1}^{i-1} \frac{1}{2^r-1} \times \prod_{s=1}^{\ell-i} \frac{1}{1-2^{-s}}.$$

We can therefore write

$$f_\ell(x) = \sum_{i=1}^{\ell} 2e^{-2^i x} \times (-1)^{i-1} \prod_{r=1}^{i-1} \frac{2}{2^r-1} \times \prod_{s=1}^{\ell-i} \frac{1}{1-2^{-s}}.$$

This allows us to bound

$$|f_\ell(x)| \leq 2C^{-1}e^{-2x} \sum_{i=1}^{\ell} \prod_{r=1}^{i-1} \frac{2}{2^r-1},$$

where C is the constant in the statement of the lemma. Bounding the sum by a geometric series, we conclude that $|f_\ell(x)| = O(e^{-2x})$, where the bound is independent of ℓ. Applying dominated convergence, we obtain the formula in the statement of the lemma. □

Armed with this information, we can finally estimate $\Pr[\mathbf{k} < k]$.

Lemma 2.5. *The following holds for every $k \geq 1$:*

$$\Pr[\mathbf{k} < k] = \Pr\left[\mathbf{H} \geq \frac{n}{2^k}\right] \pm O\left(\sqrt{\frac{k}{2^k}}\right).$$

Proof. Since $\mathbf{H}$ has bounded density by Lemma 2.4, we can apply Lemma 1.3 to bound the Kolmogorov–Smirnov distance between $\frac{n}{2^k}\mathbf{G}$ and $\mathbf{H}$ by $O(\sqrt{W_1(\frac{n}{2^k}\mathbf{G}, \mathbf{H})}) = O(\sqrt{k/2^k})$, using Lemma 2.3. It follows that

$$\Pr[\mathbf{k} < k] = \Pr\left[\frac{n}{2^k}\mathbf{G} \geq \frac{n}{2^k}\right] = \Pr\left[\mathbf{H} \geq \frac{n}{2^k}\right] \pm O\left(\sqrt{\frac{k,}{2^k}}\right).$$ □

Theorem 1.2 now easily follows:

Proof of Theorem 1.2. Lemma 2.5 shows that for each $k \geq 1$,

$$\Pr[\mathbf{k} = k] = \Pr[\mathbf{k} < k+1] - \Pr[\mathbf{k} < k] = \Pr\left[\frac{n}{2^{k+1}} \leq \mathbf{H} < \frac{n}{2^k}\right] \pm O\left(\sqrt{\frac{k}{2^k}}\right).$$

This implies that for $\ell \geq 1$,

$$\sum_{k=\ell}^{\infty} \left|\Pr[\mathbf{k} = k] - \Pr\left[\frac{n}{2^{k+1}} \leq \mathbf{H} < \frac{n}{2^k}\right]\right| = O\left(\sqrt{\frac{\ell}{2^\ell}}\right).$$

Lemma 2.1 and Markov's inequality show that

$$\Pr[\mathbf{k} < \ell] = \Pr[\mathrm{G}(1/2) + \cdots + \mathrm{G}(1/2^{\ell-1}) \geq n] \leq \frac{\mathbb{E}[\mathrm{G}(1/2) + \cdots + \mathrm{G}(1/2^{\ell-1})]}{n} < \frac{2^\ell}{n},$$

and so choosing $\ell := \frac{2}{3}\log_2 n$, we have

$$\Pr[\mathbf{k} < \ell] \leq \frac{1}{n^{1/3}}.$$

Since $\sqrt{\ell/2^\ell} = O(\sqrt{\log n}/n^{1/3})$, Lemma 2.5 shows that

$$\Pr\left[\mathbf{H} \geq \frac{n}{2^\ell}\right] = O\left(\frac{\sqrt{\log n}}{n^{1/3}}\right),$$

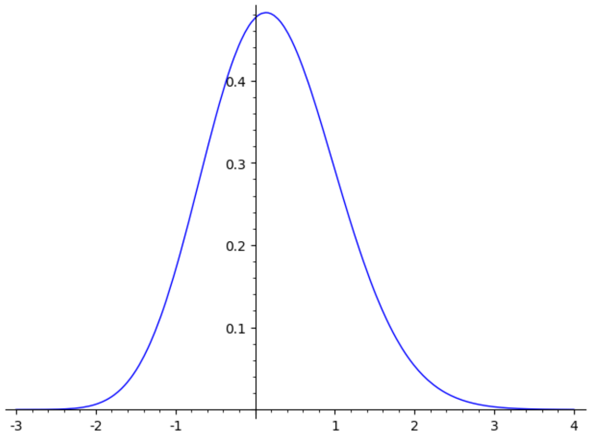

Figure 1: Density of $\log_2(1/\mathbf{H})$

and so

$$\sum_{k=0}^{\ell-1} \left|\Pr[\mathbf{k}=k] - \Pr\left[\frac{n}{2^{k+1}} \le \mathbf{H} < \frac{n}{2^k}\right]\right| \le \sum_{k=0}^{\ell-1} \left(\Pr[\mathbf{k}=k] + \Pr\left[\frac{n}{2^{k+1}} \le \mathbf{H} < \frac{n}{2^k}\right]\right) = O\left(\frac{\sqrt{\log n}}{n^{1/3}}\right).$$

In total, we conclude that

$$\sum_{k=0}^{\infty} \left|\Pr[\mathbf{k}=k] - \Pr\left[\frac{n}{2^{k+1}} \le \mathbf{H} < \frac{n}{2^k}\right]\right| = O\left(\frac{\sqrt{\log n}}{n^{1/3}}\right). \qquad \square$$

We can also express Theorem 1.2 in terms of the variation distance between $\mathbf{k}$ and an appropriate random variable.

Let $\theta = \{\log_2 n\} = \log_2 n - \lfloor \log_2 n \rfloor$, and let $k = \lfloor \log_2 n \rfloor + c$. Then $n/2^k = 2^{\theta - c}$, and so the quantity q_k in Theorem 1.2 is

$$\Pr[2^{-(c+1)} \le 2^{-\theta}\mathbf{H} < 2^{-c}] = \Pr[2^{-(c+1)} < 2^{-\theta}\mathbf{H} \le 2^{-c}] = \Pr[\lfloor \log_2(1/\mathbf{H}) + \theta \rfloor = c].$$

Therefore we obtain the following corollary:

Corollary 2.6. *For a given n, let $\theta = \{\log_2 n\}$ and define*

$$\mathbf{h} = \lfloor \log_2(1/\mathbf{H}) + \theta \rfloor.$$

The variation distance between $\mathbf{k}$ and $\mathbf{h}$ is at most $\tilde{O}(1/n^{1/3})$.

The random variable $\log_2(1/\mathbf{H})$ has density

$$g(y) = (2C^{-1}\ln 2)2^{-y} \sum_{i=1}^{\infty} (-1)^{i-1} e^{-2^{i-y}} \prod_{r=1}^{i-1} \frac{2}{2^r - 1},$$

and is plotted in Fig. 1.

3 Applications

Integrating the formula given in Lemma 2.4, we obtain the following estimate via Lemma 2.5:

$$Pr[\mathbf{k}=k] \approx C^{-1} \sum_{i=1}^{\infty} (-1)^{i-1} \left(e^{-n2^{i-k-1}} - e^{-n2^{i-k}}\right) \prod_{r=1}^{i-1} \frac{1}{2^r - 1},$$

where the error is $O(k/2^k)$. If $k = \log_2 n + c$, then this becomes

$$Pr[\mathbf{k} = \log_2 n + c] \approx C^{-1} \sum_{i=1}^{\infty} (-1)^{i-1} \left(e^{-2^{i-c-1}} - e^{-2^{i-c}} \right) \prod_{r=1}^{i-1} \frac{1}{2^r - 1}.$$

Using this, we can calculate the limiting distribution of $\mathbf{k}$, fixing $\{\log_2 n\}$. For example, if n is a power of 2 then we obtain the following limiting distribution:

c	$\lim \Pr[\mathbf{k} = \log_2 n + c]$
-4	0.000000389680708123307
-3	0.00116084271918975
-2	0.0610996920580558
-1	0.343335642221465
0	0.420730421531672
1	0.153255882765631
2	0.0194547690538043
3	0.000943671851018291
4	0.0000185343323798604
5	0.000000153237063593714

In this case, the expected deviation of $\mathbf{k}$ from $\log_2 n$ is -0.273947769982407, and the standard deviation of $\mathbf{k}$ is 0.763009254799132.

References

[BBPP99] Immanuel M. Bomze, Marco Budinich, Panos M. Pardalos, and Marcello Pelillo. The maximum clique problem. In *Handbook of combinatorial optimization, Supplement Vol. A*, pages 1–74. Kluwer Acad. Publ., Dordrecht, 1999.

[BC96] Mark Brockington and Joseph C. Culberson. Camouflaging independent sets in quasi-random graphs. In *Cliques, coloring, and satisfiability (New Brunswick, NJ, 1993)*, volume 26 of *DIMACS Ser. Discrete Math. Theoret. Comput. Sci.*, pages 75–88. Amer. Math. Soc., Providence, RI, 1996.

[BE76] B. Bollobás and P. Erdős. Cliques in random graphs. *Math. Proc. Cambridge Philos. Soc.*, 80(3):419–427, 1976.

[Bol01] Béla Bollobás. *Random graphs*, volume 73 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2001.

[BT85] Béla Bollobás and Andrew Thomason. Random graphs of small order. In *Random graphs '83 (Poznań, 1983)*, volume 118 of *North-Holland Math. Stud.*, pages 47–97. North-Holland, Amsterdam, 1985.

[Col64] A. J. Cole. The preparation of examination time-tables using a small-store computer. *The Computer Journal*, 7(2):117–121, 01 1964.

[COT04] Amin Coja-Oghlan and Anusch Taraz. Exact and approximative algorithms for coloring $G(n, p)$. *Random Structures Algorithms*, 24(3):259–278, 2004.

[CR07] Sourav Chatterjee and David Rosenberg. Stein's method and applications. Lecture notes, 2007.

[Fel71] William Feller. *An introduction to probability theory and its applications. Vol. II.* Second edition. John Wiley & Sons, Inc., New York-London-Sydney, 1971.

[FK16] Alan Frieze and MichałKaroński. *Introduction to random graphs.* Cambridge University Press, Cambridge, 2016.

[Fla85] Philippe Flajolet. Approximate counting: a detailed analysis. *BIT*, 25(1):113–134, 1985.

[FM97] Alan Frieze and Colin McDiarmid. Algorithmic theory of random graphs. volume 10, pages 5–42. 1997. Average-case analysis of algorithms (Dagstuhl, 1995).

[FMS96] James Allen Fill, Hosam M. Mahmoud, and Wojciech Szpankowski. On the distribution for the duration of a randomized leader election algorithm. *Ann. Appl. Probab.*, 6(4):1260–1283, 1996.

[GM75] G. R. Grimmett and C. J. H. McDiarmid. On colouring random graphs. *Math. Proc. Cambridge Philos. Soc.*, 77:313–324, 1975.

[Hec20] Annika Heckel. Non-concentration of the chromatic number of a random graph. *Journal of the American Mathematical Society*, 2020. To appear.

[Hwa08] Hsien-Kuei Hwang. Probabilistic analysis of an exhaustive search algorithm in random graphs. Conference slides, 2008.

[Jer92] Mark Jerrum. Large cliques elude the Metropolis process. *Random Structures Algorithms*, 3(4):347–359, 1992.

[JLL08] Svante Janson, Christian Lavault, and Guy Louchard. Convergence of some leader election algorithms. *Discrete Math. Theor. Comput. Sci.*, 10(3):171–196, 2008.

[JŁR00] Svante Janson, Tomasz Łuczak, and Andrzej Ruciński. *Random graphs*. Wiley-Interscience Series in Discrete Mathematics and Optimization. Wiley-Interscience, New York, 2000.

[Joh74] David S. Johnson. Worst case behavior of graph coloring algorithms. In *Proceedings of the Fifth Southeastern Conference on Combinatorics, Graph Theory and Computing (Florida Atlantic Univ., Boca Raton, Fla., 1974)*, pages 513–527. Congressus Numerantium, No. X, 1974.

[Kar76] Richard M. Karp. The probabilistic analysis of some combinational search algorithms. Technical Report UCB/ERL M581, EECS Department, University of California, Berkeley, Apr 1976.

[KM15] Ross J. Kang and Colin McDiarmid. Colouring random graphs. In *Topics in chromatic graph theory*, volume 156 of *Encyclopedia Math. Appl.*, pages 199–229. Cambridge Univ. Press, Cambridge, 2015.

[Kne01] Charles Knessl. Asymptotic and numerical studies of the leader election algorithm. *European J. Appl. Math.*, 12(6):645–664, 2001.

[Kri02] Michael Krivelevich. Coloring random graphs—an algorithmic perspective. In *Mathematics and computer science, II (Versailles, 2002)*, Trends Math., pages 175–195. Birkhäuser, Basel, 2002.

[KS98] Michael Krivelevich and Benny Sudakov. Coloring random graphs. *Inform. Process. Lett.*, 67(2):71–74, 1998.

[Kuč77] Luděk Kučera. Expected behavior of graph coloring algorithms. In *Fundamentals of computation theory (Proc. Internat. Conf., Poznań-Kórnik, 1977)*, pages 447–451. Lecture Notes in Comput. Sci., Vol. 56, 1977.

[Kuč92] Luděk Kučera. A generalized encryption scheme based on random graphs. In *Graph-theoretic concepts in computer science (Fischbachau, 1991)*, volume 570 of *Lecture Notes in Comput. Sci.*, pages 180–186. Springer, Berlin, 1992.

[Kuč95] Luděk Kučera. Expected complexity of graph partitioning problems. volume 57, pages 193–212. 1995. Combinatorial optimization 1992 (CO92) (Oxford).

[KV02] Michael Krivelevich and Van H. Vu. Approximating the independence number and the chromatic number in expected polynomial time. *J. Comb. Optim.*, 6(2):143–155, 2002.

[LP08] Guy Louchard and Helmut Prodinger. Advancing in the presence of a demon. *Math. Slovaca*, 58(3):263–276, 2008.

[Mat68] David W. Matula. A min-max theorem for graphs with application to graph coloring. *SIAM Review*, 10:481–482, 1968.

[Mat72] David W. Matula. The employee party problem. *Not. Amer. Math. Soc.*, 19(2):A–382, 1972.

[Mat76] David W. Matula. The largest clique size in a random graph. Technical Report 7608, Department of Computer Science, Southern Methodist University, 1976.

[McD79] Colin McDiarmid. Colouring random graphs badly. In *Graph theory and combinatorics (Proc. Conf., Open Univ., Milton Keynes, 1978)*, volume 34 of *Res. Notes in Math.*, pages 76–86. Pitman, Boston, Mass.-London, 1979.

[McD84] Colin McDiarmid. Colouring random graphs. *Annals of Operations Research*, 1(3):183–200, 1984.

[MMI72] David W. Matula, George Marble, and Joel D. Isaacson. Graph coloring algorithms. In *Graph theory and computing*, pages 109–122. 1972.

[Mor78] Robert Morris. Counting large numbers of events in small registers. *Commun. ACM*, 21(10):840–842, October 1978.

[Pal88] Zbigniew Palka. Asymptotic properties of random graphs. *Dissertationes Math. (Rozprawy Mat.)*, 275:112, 1988.

[Pro92] Helmut Prodinger. Hypothetical analyses: approximate counting in the style of knuth, path length in the style of flajolet. *Theoretical Computer Science*, 100(1):243–251, 1992.

[Pro93a] Helmut Prodinger. How to advance on a stairway by coin flippings. In *Applications of Fibonacci numbers, Vol. 5 (St. Andrews, 1992)*, pages 473–479. Kluwer Acad. Publ., Dordrecht, 1993.

[Pro93b] Helmut Prodinger. How to select a loser. *Discrete Math.*, 120(1-3):149–159, 1993.

[PZ20] Victor M. Panaretos and Yoav Zemel. *An Invitation to Statistics in Wasserstein Space*. Springer International Publishing, 2020.

[WP67] D. J. A. Welsh and M. B. Powell. An upper bound for the chromatic number of a graph and its application to timetabling problems. *The Computer Journal*, 10(1):85–86, 01 1967.