Spectral Methods in Extremal Combinatorics

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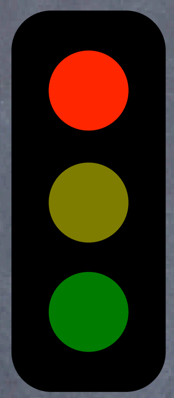
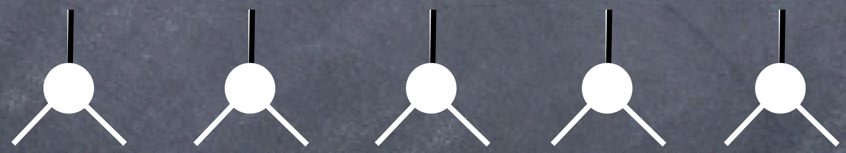
A traffic light puzzle

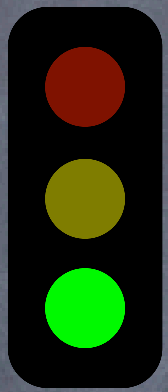


- Traffic light is controlled by 3-way switches

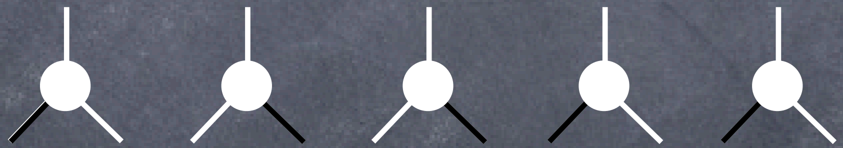


- When all switches change, light changes

A traffic light puzzle



- Traffic light is controlled by 3-way switches



- When all switches change, light changes
- Show that light is controlled by one switch!

Outline of the talk

- **Solution of the puzzle using Hoffman's bound**
- Triangle-intersecting families of graphs
- Structure theorems for functions on S_n
- Applications to theoretical computer science

Solution of the puzzle

- R = set of switch settings leading to red
- No two x,y∈R differ on <u>all</u> switches
- R is an independent set in a graph G=(V,E):
  - V is set of all switch settings
  - (x,y)∈E if they differ on all switches

Hoffman's bound

- Hoffman [1969]: (similar to Lovász θ function)
  Let A be adjacency matrix of d-regular graph G, λ_{min} smallest eigenvalue of A:
  - $\alpha(G) \leq -\lambda_{min}/(d-\lambda_{min})\ |V|$
  - If $\alpha(G) = -\lambda_{min}/(d-\lambda_{min})\ |V|$ then:
    char. func. of any maximum independent set is in linear span of **1** and eigenspace of λ_{min}

Solution of the puzzle

- Hoffman's bound: $\alpha(G) \leq -\lambda_{min}/(d-\lambda_{min})\ |V|$
- In our case $d = 2^n$, $\lambda_{min} = -2^{n-1}$, $|V| = 3^n$, so $\alpha(G) \leq 3^{n-1}$
- Since $|R|,|Y|,|G| \leq 3^{n-1}$, $|R| = |Y| = |G| = 3^{n-1}$
- Equality part of Hoffman's bound implies: Each of R,Y,G depends on only one switch

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Triangle-intersecting families of graphs

- Family of graphs on [n] is triangle-intersecting if intersection of any two graphs contains a triangle
- Simonovits and Sós [1976]: How large can family be?
  - Conjecture: at most 1/8 of all graphs
- Chung, Frankl, Graham, Shearer [1986]: at most 1/4
- Ellis, F., Friedgut [2012]: verify the conjecture

Weighted Hoffman's bound

- Hoffman's bound works on edge-weighted graphs
- Weighted vertex degree must be constant
- We construct a weighted graph $G=(V,E,w)$:
  - V = all graphs on $[n]$
  - $(x,y)\in E$ if $x\cap y$ is triangle-free
  - w is engineered to give bound $|V|/8$

Weighted Hoffman's bound

- Hoffman's bound works on edge-weighted graphs
- Weighted vertex degree must be constant
- We construct a weighted graph G=(V,E,w):
  - V = all graphs on [n]
  - $(x,y)\in E$ if $\overline{x\Delta y}$ is bipartite
  - w is engineered to give bound |V|/8

Uniqueness & Stability

- Uniqueness:
  - Char. func. of a maximum independent set is linear comb. of functions depending on 3 edges
  - Implies that any maximum independent set is triangle-junta (all supergraphs of fixed triangle)
- Stability:
  - What about ind. sets of almost maximal size?

Uniqueness & Stability

- What about ind. sets of almost maximal size?
- Suppose F contains (1−ε) |V|/8 graphs
- Hoffman's bound: 1_F close to linear combination of functions depending on 3 edges
- Kindler-Safra [2002]: F close to a family G depending on O(1) edges
- ε small enough ⇒ G is triangle-junta

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Intersecting families of permutations

- Family of permutations in S_n is <u>intersecting</u> if any two σ,τ agree on some point: $\exists i\ \sigma(i) = \tau(i)$
- Easy: size of max. intersecting family = $(n-1)!$
- Hard: maximum intersecting families are

$$T_{ij} = \{ \sigma \in S_n : \sigma(i) = j \}$$

- What about families of size $(1-\varepsilon)(n-1)!$?

Using Hoffman's method

- Let $G=(V,E)$ where $V = S_n$ and $(\sigma,\tau)\in E$ if σ,τ differ on all points, i.e. $\sigma\tau^{-1}$ is a derangement
- Renteln [2007] calculated $\lambda_{min} = -d_n/(n-1)$
- Hoffman's bound: $\alpha(G) \leq (n-1)!$ and maximum independent set in linear span of T_{ij}
  - Implies maximum independent sets are T_{ij}

Stability

- Suppose F contains (1−ε) (n−1)! permutations
- Hoffman's bound: 1_F close to linear comb. of T_{ij}
- Ellis, Friedgut & Pilpel [2011]:
  Since F is intersecting, F is close to some T_{ij}
- Ellis, F., Friedgut [2012]:
  Similar result without assuming F is intersecting

Our structure theorems

- Suppose 1_F is close to linear combination of T_{ij}
  - Thm 1: If $|F| = c\,(n-1)!$ then F is close to a union of T_{ij}'s (works for small c)
  - Thm 2: If $|F| = c\,n!$ then F is close to a **disjoint** union of T_{ij}'s (works for c far from 0,1)
    So $F \approx \{\sigma\in S_n : \sigma(i)\in J\}$ or $F \approx \{\sigma\in S_n : \sigma^{-1}(j)\in I\}$
- Thm 1 & Thm 2 have completely different proofs

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Applications

- Our results don't have applications to theoretical computer science yet
- Similar results have many applications:
  - Voting theory (quantitative Arrow's theorem)
  - Inapproximability (vertex cover)
  - Property testing (assignment testers)

Any questions?