# Riddle Concerning $\pm 1$ Vectors

Yuval Filmus

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*You are given the list of all $2^n$ vectors with $\pm 1$ entries of length $n$. Someone erased (changed to $0$) some of the entries. Show that there's always a non-empty set of rows (vectors) summing to the zero vector.*

Each $\pm 1$ vector is the difference of two $0/1$ vectors, for example:

$$\begin{array}{rrrrr} & 1 & -1 & 1 & -1 \\ = & 1 & 0 & 1 & 0 \\ - & 0 & 1 & 0 & 1 \end{array}$$

In this representation, every $0/1$ vector appears exactly once as a minuend and once as a subtrahend.

We can handle erasure of entries by modifying all subtrahends. If a $1$ is erased, we modify the corresponding entry in the subtrahend from $0$ to $1$. If a $-1$ is erased, we modify the corresponding entry from $1$ to $0$. Continuing our previous example:

$$\begin{array}{rrrrr} & 0 & 0 & 1 & -1 \\ = & 1 & 0 & 1 & 0 \\ - & 1 & 0 & 0 & 1 \end{array}$$

Let us number the lines according to the minuends. Define $f(x)$ to be the subtrahend at line $x$, so that line $x$ is equal to $\ell(x) = x - f(x)$. The function $f$ maps $\mathbb{Z}_2^n$ to itself, and so if we apply it repetitively to some element, eventually it will reach a cycle $x_0, \ldots, x_{m-1}$, i.e. $f(x_i) = x_{i+1 \pmod m}$. The lines $x_0, \ldots, x_{m-1}$ sum to zero:

$$\sum_{i=0}^{m-1} \ell(x_i) = \sum_{i=0}^{m-1} x_i - f(x_i) = \sum_{i=0}^{m-1} x_i - x_{i+1 \pmod m} = 0.$$