Products of Geometric Series

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April 2005

AK asked the following question: Give a combinatorial proof for the identity

$$\frac{1}{1-x-y+xy} = \sum x^i y^j.$$

Note that an analytic proof is very easy, since

$$\frac{1}{1-x-y+xy} = \frac{1}{1-x} \cdot \frac{1}{1-y}.$$

Considering both sides as geometric series, we give a combinatorial proof of

$$\sum (x+y-xy)^n = \sum x^i y^j.$$

We consider the left-hand side as weighted paths from the origin to points of the integer lattice. Each path is a sequence of arrows of one of three types, x, y and $[xy]$. The weight of the first two is $+1$, and the weight of the last is -1. The weight of a path is simply the product of the weights of its arrows. The identity simply says the the sum of weights of all paths from $(0,0)$ to (i,j) is 1. We shortly present a sign-reversing bijection between $(0,0)-(i,j)$ paths, which is defined for all but a single path of weight $+1$. This proves the identity.

The bijection can be stated by the following simple rule:

$$xy \leftrightarrow [xy].$$

This rule tells us to follow the path until the first occurrence of either substring, and then follow the rule. Evidently this switches the sign, and the the rule fails only for the path $y^j x^i$.

The proof can be generalized to products of more than two series. In order to prove

$$\prod \frac{1}{1-x_i} = \sum x_i^{\alpha_i},$$

we follow the rules of the form

$$x_{i_1}[x_{i_2} \cdots x_{i_m}] \leftrightarrow [x_{i_1} \cdots x_{i_m}], \quad x_{i_1} < \cdots < x_{i_m}.$$

The only path remaining is then $x_n^{\alpha_n} \cdots x_1^{\alpha_1}$.

For example, if $n=3$ then the rules take the following form:

$$\begin{aligned} x[yz] &\leftrightarrow [xyz], \\ xy &\leftrightarrow [xy], \\ xz &\leftrightarrow [xz], \\ yz &\leftrightarrow [yz]. \end{aligned}$$

The only remaining path is $z^k y^j x^i$.