# Dehn's Theorem Yet Again

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September 2006

In this writeup we shall prove Dehn's theorem 'intuitively'. We modify the proof via homomorphism so that the resulting operation is a simple grid change (without negative values), and adapt the proof by using a more sophisticated notion of area.

The main lemma is as follows.

**Lemma 1** *For each finite set of positive reals there exists a base (over $\mathbb{Q}$) consisting of positive reals such that each element of the set is a non-negative (even integral) combination of base elements.*

**Proof.** Denote the set $S$, and pick some base $B$ for it. By possibly negating elements of $B$, we can assume that $B$ is positive (consists of positive elements). We go over the elements of $S$ one after another, making sure that in each stage $B$ remains positive and all the elements have a positive representation. Clearly this applies when no element has been processed.

We now show how to process an element $s \in S$, keeping the induction hypothesis. We consider the representation of all processed elements (including $s$) by $B$ as a matrix $M$, such that when all elements will have been processed we would have $S = MB$. Currently all but the last line of $M$ is non-negative. Since $s > 0$, in the representation of $s$ by $B$ there is at least one positive coefficient, say of $b \in B$. In other words, the last line contains a positive element. Now consider the operation of adding one column to another, suppose the one corresponding to $\alpha \in B$ to the one corresponding to $\beta \in B$. Clearly this amounts to replacing $\alpha$ with $\alpha - \beta$. Thus we can add the column corresponding to $b$ with sufficient multiplicity to all other columns, ensuring $M$ is positive. This changes only $b$, which can turn out to be negative. We now show how to correct this.

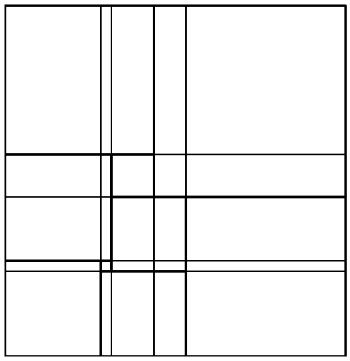
Suppose then that $M$ is positive but one (and only one) of the basis elements, say $b \in B$, is negative. We replace $b$ by $-b$ in $B$, and now $B$ is positive and $M$ contains a negative column. Without loss of generality, we can assume that all entries of this column are $-1$. We want to make the negative column positive by adding to it a combination of other columns. This amounts to inequalities of the form $\sum \alpha_i c_i > 1$, where $\alpha_i$ is the combination, and $c_i$ is a row of $M$. Since this row is a representation of a positive element, we have $\sum b_i c_i > b$. The resulting basis will be positive given $b_i > \alpha_i b$, that is $\alpha_i < b_i/b$. Choosing $\alpha_i = b_i/b - \varepsilon_i$ where we think of $\varepsilon_i > 0$ as 'small' will work if for each row $c_i$ we would have $\sum b_i c_i > b + \sum b c_i \varepsilon_i$. Since $\sum b_i c_i > b$ it is clear that choosing small enough $\varepsilon$'s will work. Since the number of constraints is finite, we can choose $\varepsilon$'s such that both $M$ and the resulting base $B$ will be positive. $\square$

Here is a simpler proof, by Avinoam Braverman.

**Proof.** We join the previous proof when adding a new element $s \in S$. The last line contains the representation of $s$ in terms of $B$. Without loss of generality, we can assume that all the non-negative entries are $\pm 1$. As before, adding one column (corresponding to $\alpha$) to another (corresponding to $\beta$) changes $\alpha$ to $\alpha - \beta$. This is legal (keeps the basis positive) if $\alpha > \beta$. Thus, given any two columns, one of them can be added to the other one keeping the basis positive. Now iteratively pick any two non-zero columns of opposite in the last line, and add one of them to the other. At each step the number of non-zero columns decreases, so finally we will reach a state where all non-zero columns have the same sign. This sign, however, must be positive, since the basis is positive and the element $s$ represented by the line is positive. $\square$

We next recall the homomorphism proof of Dehn's theorem, and show how the lemma helps us make the proof more 'intuitive'.

The proof works by first extending all lines in the diagram (see the figure below).



We now collect all the resulting side lengths into a set $S$, and find a base $B$ for it over $\mathbb{Q}$. For any bilinear form $\Lambda$ over $\mathbb{Q}$ taking both side lengths of a rectangle, it is clear that $\Lambda\Box = \sum \lambda\Box_i$, where $\Box$ is the big rectangle and $\Box_i$ are the smaller ones (this is evident if we consider the subpartition formed by the line extension). Picking $\Lambda(x, y) = \varphi(x)\varphi(y)$, we have $\lambda\Box_i = \lambda(x_i)^2 \geq 0$, whereas $\Lambda\Box$ can be chosen to be negative if the sides of the rectangle are not equivalent over $\mathbb{Q}$. The homomorphism $\varphi$ amounts to a change of side lengths, which may change some lengths to be negative. The resulting concept of area is thus signed. Using our lemma, we can overcome this difficulty by making sure that all the resulting lengths are actually positive, and letting $\varphi$ assign positive values to all base elements. However, the proof itself becomes more complicated and less transparent.

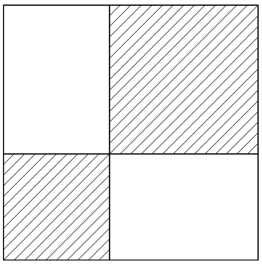
From now on assume that $B$ is chosen according to the lemma. Denote by $x, y$ the sides of the large rectangle. The easy case is when there exists $b \in B$ such that the coefficient of $b$ in $x$ is zero, and the corresponding coefficient in $y$ is positive. In this case we let $\varphi(b) = M$ and $\varphi(b_i) = \varepsilon$ for all other base elements, where we think of $M$ as 'large' and of $\varepsilon$ as 'small'. Clearly for each square either $\Lambda = \Theta(\varepsilon^2)$ or $\Lambda = \Theta(M^2)$, whereas for the big rectangle $\Lambda = \Theta(M)$, contradiction.

We now tackle the general case. Take two base elements $b$ and $c$ which appear non-trivially in both $x$ and $y$, and forget for the moment all other elements. Let $\varphi$ take $b$ to $M$ and $c$ to $N$. Denote $x_i = \beta_i b + \gamma_i c$, $x = \beta_x b + \gamma_x c$, $y = \beta_y b + \gamma_y c$. Thus $\Lambda_i = \beta_i^2 M^2 + \gamma_i^2 N^2 + 2\beta_i\gamma_i MN$, and $\Lambda = \beta_x\beta_y M^2 + \gamma_x\gamma_y N^2 + (\beta_x\gamma_y + \beta_y\gamma_x)MN$. Recall that $\Lambda = \sum \Lambda_i$. The bilinear forms $\Lambda_i$ all satisfy $\lambda^2 A + \mu^2 B \geq \lambda\mu C$ for all $\lambda, \mu$, where the form is $AM^2 + BN^2 + CMN$. The same should be satisfied by $\Lambda$, that is

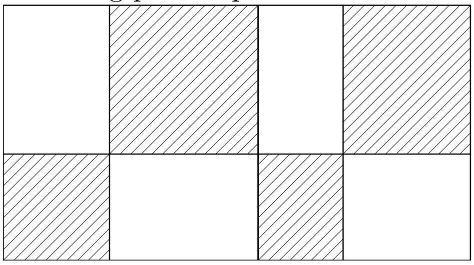
$$(\lambda\beta_x)(\lambda\beta_y) + (\mu\gamma_x)(\mu\gamma_y) \geq (\lambda\beta_x)(\mu\gamma_y) + (\lambda\beta_y)(\mu\gamma_x)$$

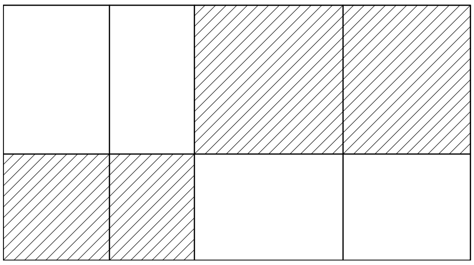
for all choices of $\lambda, \mu$. Looking at this as a bilinear form in $\lambda, \mu$, the discriminant must be non-positive, that is $(\beta_x\gamma_y + \gamma_x\beta_y)^2 \leq 4\beta_x\beta_y\gamma_x\gamma_y$, which simplifies to $(\beta_x\gamma_y - \gamma_x\beta_y)^2 \leq 0$. Hence $\beta_x\gamma_y = \gamma_x\beta_y$ so that $x$ and $y$ are linearly dependent.

We proceed to give a graphic interpretation of the above proof. Consider any rectangle with side lengths $\beta_x b + \gamma_x c$ and $\beta_y b + \gamma_y c$. We divide the area of the rectangle into two parts: the *square* part (marked below) and the *rectangle* parts.



Both parts are additive, as the following picture proof shows.





It is easy to see that for each square, the square part is at least as large as the rectangle part. A graphic proof can be given by providing a graphic interpretation for the equation $x^2 + y^2 = 2xy + (x - y)^2$. We proceed to show that a proper choice of homomorphism makes this false for the encompassing rectangle.

Without loss of generality we can assume that $x = N + M$ and $y = N + cM$ for $c > 1$. Choose a $\varphi$ that sends $N$ to 1 and $M$ to $1 - \varepsilon$. The square part is thus $1 + (1 - \varepsilon)^2 c$ and the rectangle part is $(1 - \varepsilon)(1 + c)$, which is bigger for small $\varepsilon$ (check! consider $\varepsilon = 0$).