Random Segments on a Circle

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1 Introduction

Suppose n points are thrown on the circumference of a circle of unit length. The n points create n segments. What is the expected length of the smallest segment? The well known answer is $1/n^2$. In general, the expected length of the kth largest segment is

$$\frac{1}{n}\left(\frac{1}{n}+\cdots+\frac{1}{k}\right).$$

Elfi now asked the question — suppose we look at pairs of adjacent segments. What is the expected length of the smallest/largest such pair? In particular, does there exist a nice formula for it?

Attempts to generalize Elfi's trick on the classical problem (removing the smallest segment from each segment, thus reducing the number of points) failed. In this note we show how to calculate the expected length, and provide several numerical values. From these values it becomes apparent that no simple formula exists for our new problem.

2 Calculation for $n = 3$

We start with the simple case $n = 3$. In this case we know that the expected value of the smallest pair is equal to one less than the expected value $1/3(1/3+1/2+1) = 11/18$ of the largest single. This will enable us to verify our calculations.

Denote the (circularly ordered) points by P, Q, R. We can assume that PR is the smallest pair, and moreover can center the points such that $P = 0$. We thus rename the points to $0, a, b$. The fact that the points are ordered is expressed by the inequality $0 \leq a \leq b \leq 1$. Moreover, the lengths of the pairs

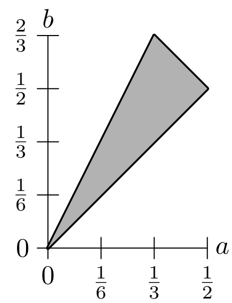

Figure 1: Region for $n = 3$

are $b, 1 - a, 1 - b + a$, and so the fact that b is the endpoint of the smallest pair is expressible by the following system of inequalities:

$$0 \leq a \leq b \leq 1,$$
$$b \leq 1 - a, 1 - b + a.$$

Now suppose a point is picked up uniformly at random inside the triangle bound by these inequalities (see the figure). The expected length of the smallest pair is thus the expected value of b.

How do we calculate the expected value of b? We shall accomplish that by presenting the region as a a disjoint sum of regions of the following type:

$$\alpha_0 \leq a \leq \alpha_1,$$
$$\alpha_2 a + \beta_0 \leq b \leq \alpha_3 a + \beta_1.$$

It is easy to calculate $\int b$ over such a region. The required expected value is now given by $\sum \int b / \sum \int 1$, where the denominator is needed since our choices of $P = 0$ and PR as the smallest pair mean that the points a, b are no longer uniformly chosen on the unit circle. The volume $\int 1$ should equal $1/6$ by our two choices (check!).

How do we present the region thus? We will start with a trivial region, and add new inequalities one by one. Our starting point will be

$$0 \leq a \leq 1, \ 0 \leq b \leq 1.$$

We will add the following inequalities: $a \leq b$, $b \leq 1-a$ and finally $b \leq 1-b+a$. Adding $a \leq b$ poses no problem, and we get

$$0 \leq a \leq 1, \ a \leq b \leq 1.$$

Adding $b \leq 1 - a$ changes the second inequality to $a \leq b \leq 1 - a$, which limits a to $[0, 1/2]$, and thus we are left with

$$0 \leq a \leq \frac{1}{2}, \ a \leq b \leq 1 - a.$$

The last inequality is also the most difficult. Let us put it in a more appropriate form: $b \leq (1+a)/2$. There are two cases: either $a \leq 1/3$, in which case $(1+a)/2 \leq 1-a$, or $a \geq 1/3$, in which case the reverse inequality is true. We are led to split our one region into two:

$$0 \leq a \leq \frac{1}{3}, a \leq b \leq \frac{1+a}{2};$$
$$\frac{1}{3} \leq a \leq \frac{1}{2}, a \leq b \leq 1-a.$$

The reader can now calculate $\sum \int b = 7/108$ and $\sum \int 1 = 1/6$, and so the expected length of the smallest pair is $7/18 = 1 - 11/18$, as expected. In more detail, the calculation of the numerator is

$$\int_0^{\frac{1}{3}} \int_a^{\frac{1+a}{2}} b\, db\, da + \int_{\frac{1}{3}}^{\frac{1}{2}} \int_a^{1-a} b\, db\, da.$$

3 The Solver

In this section we shall generalize the method by which the case $n = 3$ was "solved". We shall show how to produce, given a system of linear inequalities in bound variables, a disjoint sum of easily integrable regions. Given an order of the variables v_i, these regions are given by linear bounds on v_i which depend only on v_j for $j < i$.

Clearly it is enough to consider adding one linear inequality to a single easily integrable region. Denote the region by $l_i \leq v_i \leq r_i$, and the inequality by $E \geq 0$. Suppose that v_i is the largest variable appearing in E, and suppose the coefficient of v_i in E is -1 (the case $+1$ is symmetric). We can thus rewrite the inequality as $v_i \leq B$. We first consider whether it is always true that $B \leq r_i$ or $r_i \leq B$; we will show later how. If one of these is true, it will be enough to consider only one of the cases below.

Suppose for now that neither $B \leq r_i$ nor $r_i \leq B$ is always true. We have to split our region into two. The first part is obtained by assuming that $B \leq r_i$. We thus replace r_i by B, and add the inequality $B \leq r_i$ to the resulting region recursively. The second part is obtained by assuming that $r_i \leq B$. In that case we do not replace r_i, but we do add $r_i \leq B$ recursively. Note that if we already know that always $B \leq r_i$ or $r_i \leq B$ then there is no need to add this inequality.

How do we know whether (for example) $B \leq r_i$? Let's rewrite it $0 \leq r_i - B$. We now replace the maximal variable v_j appearing in $r_i - B$ by either l_j or r_j, according to the sign (positive or negative, respectively). We continue until all variables are eliminated, and we check whether the resulting inequality is of the form $0 \leq c$ for non-negative c.

4 kth Smallest Pair

We now show how to solve the original problem. Finding the smallest pair is done by generalizing the system of inequality given above for the case $n = 3$:

$$0 \leq v_1 \leq \cdots \leq v_{n-1} \leq 1,$$
$$v_2 \leq v_3 - v_1, \ldots, v_{n-1} - v_{n-3}, 1 - v_{n-2}, 1 - v_{n-1} + v_1.$$

Finding the largest pair is facilitated by changing the direction of the second row of inequalities.

In order to find the second smallest pair, we go over all possible relative locations of the smallest and second smallest pairs, and then write inequalities as above. The details are left to the reader.

5 Results

The following table presents several expectations. Here n is the number of points and k is the relative order of the pair.

n,k	1	2	3	4	5	6
3	7/18	13/18	8/9			
4	1/4	3/8	5/8	3/4		
5	38/225	121/450	649/1800	41/75	131/200	
6	1/8	125/648	175/648	55/162	317/648	7/12

The following table is for triplets.

n,k	1	2	3	4	5	6
4	23/48	35/48	41/48	15/16		
5	69/200	34/75	1151/1800	329/450	187/225	
6	19/72	25/72	121/288	167/288	47/72	53/72