Combinatorial Games

Solution to exercise 3.7.6

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**Notation.** Throughout, $G = (V, E)$ will be a finite digraph, and $\gamma$ will be its (unique) $\gamma$-function. A value function $g$ is any function $g\colon V \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$. The notation $g(v) = \infty(K)$ means that $K = g(F(v))$, where $F(v) = \{u : (v, u) \in E\}$. Also, $F^{-1}(v) = \{u : (u, v) \in E\}$. We define

$$g'(v) = \operatorname{mex} g(F(v)).$$

We now define several functions which take a value function as an optional argument. When the argument is not given, $\gamma$ is understood.

$$\begin{aligned}
V^f(g) &= \{v \in V : g(v) < \infty\},\\
V^\infty(g) &= \{v \in V : g(v) = \infty\};\\
\mathbb{P}(g) &= \{v \in V : g(v) = 0\},\\
\mathbb{D}(g) &= \{v \in V : g(v) = \infty(K) \text{ and } 0 \notin K\},\\
\mathbb{N}^f(g) &= \{v \in V : 0 < g(v) < \infty\},\\
\mathbb{N}^\infty(g) &= \{v \in V : g(v) = \infty(K) \text{ and } 0 \in K\},\\
\mathbb{N}(g) &= \mathbb{N}^f(g) \cup \mathbb{N}^\infty(g).
\end{aligned}$$

Note that $(\mathbb{P}(g), \mathbb{N}(g), \mathbb{D}(g))$ is a partition of $V$, and

$$\begin{aligned}
V^f(g) &= \mathbb{P}(g) \cup \mathbb{N}^f(g),\\
V^\infty(g) &= \mathbb{D}(g) \cup \mathbb{N}^\infty(g).
\end{aligned}$$

We will also need the analogues for two tokens:

$$\begin{aligned}
\mathbb{P}_2(g) &= \{(u, v) \in V^2 : g(u) = g(v) < \infty\},\\
\mathbb{N}_2^f(g) &= \{(u, v) \in V^2 : g(u) < g(v) < \infty \text{ or } g(v) < g(u) < \infty\},\\
\mathbb{N}_2^\infty(g) &= \{(u, v) \in V^2 : (g(u) < g(v) = \infty(K) \text{ and } g(u) \in K) \text{ or } (g(v) < g(u) = \infty(K) \text{ and } g(v) \in K)\},\\
\mathbb{N}_2(g) &= \mathbb{N}_2^f(g) \cup \mathbb{N}_2^\infty(g),\\
\mathbb{D}_2^f(g) &= \{(u, v) \in V^2 : (g(u) < g(v) = \infty(K) \text{ and } g(u) \notin K) \text{ or } (g(v) < g(u) = \infty(K) \text{ and } g(v) \notin K)\},\\
\mathbb{D}_2^\infty(g) &= \{(u, v) \in V^2 : g(u) = g(v) = \infty\},\\
\mathbb{D}_2(g) &= \mathbb{D}_2^f(g) \cup \mathbb{D}_2^\infty(g);\\
V_2^f(g) &= \mathbb{P}_2(g) \cup \mathbb{N}_2^f(g),\\
V_2^\infty(g) &= \mathbb{D}_2(g) \cup \mathbb{N}_2^\infty(g).
\end{aligned}$$

Now let us define several axioms concerning value functions. Some of them concern also a counter function $c\colon V \to \mathbb{Z}_{\geq 0}$. Two axioms concern maximality given some other axioms.

**A.** If $u \in V^f(g)$ then $g(u) = g'(u)$.

**A'.** If $u \in V^f(g)$ and $i \in [0, g(u))$ then there exists $v \in F(u)$ satisfying $g(v) = i$ and $c(v) < c(u)$.

**A".** If $u \in V^f(g)$ and $v \in F(u)$ satisfies $c(v) < c(u)$ then $g(v) \neq g(u)$.

**b.** If $u \in V^f(g)$ and $v \in F(u)$ satisfies $v = \infty(K)$ then $g(u) \in K$.

**B.** If $v \in F(u)$ satisfies $g(u) < g(v)$ then there exists $w \in F(v)$ satisfying $g(w) = g(u)$ and $c(w) < c(u)$.

**B'.** If $u \in V^f(g)$ and $v \in F(u)$ satisfies either $v \in V^\infty(g)$ or $c(v) \geq c(u)$, then there exists $w \in F(v)$ satisfying $g(w) = g(u)$ and $c(w) < c(u)$.

**C.** If $u \in V^\infty(g)$ then there exists $v \in F(u)$ satisfying $g(v) = \infty(K)$ and $g'(u) \notin K$.

**c".** The function $g(v)$ is maximal with respect to $V^\infty(g)$ (given the other axioms); that is, for any non-empty set $S \subseteq V^f(g)$, if

$$G(v) = \begin{cases} g(v) & v \notin S \\ \infty & v \in S \end{cases}$$

then $G(v)$ violates one of the other axioms.

**C".** The function $g(v)$ has the maximum $|V^\infty(g)|$ (given the other axioms); that is, for any function $G(v)$ satisfying the other axioms, $|V^\infty(G)| \leq |V^\infty(g)|$.

**M.** If $\gamma(u) < \gamma(v)$ then $c(u) < c(v)$.

Note that **C"** implies **c"**.

We define several axiom systems:

$$\begin{aligned} \Gamma' &= (\mathbf{A},\, \mathbf{b},\, \mathbf{C}), \\ \Gamma &= (\mathbf{A},\, \mathbf{B},\, \mathbf{C}), \\ \Gamma^M &= (\mathbf{A},\, \mathbf{B},\, \mathbf{C},\, \mathbf{M}), \\ \Gamma_1 &= (\mathbf{A}',\, \mathbf{A}'',\, \mathbf{B}',\, \mathbf{C}), \\ \Gamma_2' &= (\mathbf{A},\, \mathbf{C};\, \mathbf{c}''), \\ \Gamma_2 &= (\mathbf{A},\, \mathbf{C};\, \mathbf{C}''). \end{aligned}$$

Note that $\Gamma^M$ is stronger than $\Gamma$, and that $\Gamma_2'$ is weaker than $\Gamma_2$.

Axiom systems $\Gamma$, $\Gamma^M$ and $\Gamma_1$ require a counter function, which is the same throughout all the axioms. To emphasize that $c$ is used as the counter function, we write $\Gamma(c)$ and $\Gamma_1(c)$.

Axiom systems $\Gamma$ and $\Gamma^M$ are satisfied by the unique value function $\gamma$ with the counter function $c$ constructed by algorithm GSG.

Our goal is to show that there is a unique value function satisfying $\Gamma_1$, namely $\gamma$, and that there is a unique value function satisfying $\Gamma_2$, again $\gamma$.

**Axiom scheme $\mathbf{\Gamma_1}$.** To establish that $\Gamma_1$ is satisfied by a unique value function, we shall show that $\Gamma^M(c) \to \Gamma_1(c)$ and that $\Gamma_1(c) \to \Gamma(c)$.

$\Gamma^M(c) \to \mathbf{A}'$: Let $u \in V^f(g)$ and let $i \in [0, g(u))$. By **A**, there is some $v \in F(u)$ satisfying $g(v) = i$. Since $g(v) < g(u)$, **M** guarantees that $c(v) < c(u)$.

$\Gamma^M(c) \to \mathbf{A}''$: Let $u \in V^f(g)$ and let $v \in F(u)$. By **A**, $g(v) \neq g(u)$.

$\Gamma^M(c) \to \mathbf{B}'$: Let $u \in V^f(g)$ and let $v \in F(u)$. We claim that if either $v \in V^\infty(g)$ or $c(v) \geq c(u)$ then $g(v) \geq g(u)$. If $v \in V^\infty(g)$ this is clear. If $c(v) \geq c(u)$ it follows from **M**. In fact, by **A**, $g(v) > g(u)$. Hence by **B** there exists $w \in F(v)$ satisfying $g(w) = g(u)$ and $c(w) < c(u)$.

$\Gamma_1(c) \to$ **A**: Let $u \in V^f(g)$. To show that $g(u) = g'(u)$, it is enough to show that (a) for any $i \in [0, g(u))$ there exists $v \in F(u)$ satisfying $g(v) = i$; and that (b) no $v \in F(u)$ satisfies $g(v) = g(u)$. Fact (a) follows from **A'**.

To prove fact (b), let us assume that fact (b) does not hold, and let $u \in V^f(g)$ be a vertex violating fact (b). Thus there exists $v \in F(u)$ satisfying $g(v) = g(u)$. Axiom **A"** shows that $c(v) \geq c(u)$. Hence **B'** shows that there exists $w \in F(v)$ satisfying $g(w) = g(u)$ and $c(w) < c(u) \leq c(v)$. Yet $g(w) = g(v)$, violating axiom **A"**.

$\Gamma_1(c) \to$ **B**: Let $v \in F(u)$ satisfy $g(u) < g(v)$. Suppose first that $v \in V^\infty(g)$ or $c(v) \geq c(u)$. Then by **B** there exists $w \in F(v)$ satisfying $g(w) = g(u)$ and $c(w) < c(u)$. If, on the other hand, $v \in V^f(g)$ and $c(v) < c(u)$, then by **A'** there exists $w \in F(v)$ satisfying $g(w) = g(u)$ and $c(w) < c(v) < c(u)$.

**Axiom scheme $\mathbf{\Gamma_2}$.** Here the proof is more involved. Note first that since $\gamma$ satisfies **A** and **C**, a function satisfying $\Gamma_2$ exists. We will prove the following theorem about the weaker system $\Gamma_2'$.

**Theorem 1** *If $g$ satisfies $\Gamma_2'$ then $V^\infty(g) \subseteq V^\infty(\gamma)$.*

**Corollary 2** *If $g$ satisfies $\Gamma_2$ then $V^\infty(g) = V^\infty(\gamma)$.*

Now the main theorem is but a corollary to a corollary.

**Theorem 3** *If $g$ satisfies $\Gamma_2$ then $g = \gamma$.*

**Proof:** Corollary 2 shows that $V^\infty(g) = V^\infty(\gamma)$. It remains to show that $g$ and $\gamma$ are identical on $V^f(g) = V^f(\gamma)$. This is just Theorem 3.5.8 of the lecture notes, whose proof holds because $V^f(g) = V^f(\gamma)$. □

On our way to prove Theorem 1, we start with an important technical result.

**Lemma 4** *Axiom scheme $\Gamma_2'$ implies axiom **b**. In other words, $\Gamma'$ is weaker than $\Gamma_2'$.*

**Proof:** We show that if **b** does not hold then minimality (axiom **c"**) fails. Indeed, suppose that **b** fails. Thus for some $u_s \in V^f(g)$ there exists $v_s \in F(u_s)$ satisfying $v_s = \infty(K)$ and $g(u_s) \notin K$.

Call a node $u \in V_f(g) \setminus S$ essential for $v \in F^{-1}(u)$ given $S$ if $g(u) < g'(v)$ and $u$ is the only vertex $w \in F(v) \setminus S$ with $g(w) = g(u)$. We shall construct a set $S$ inductively. The starting point is $S = \{u_s\}$. As long as there is a vertex $s \in S$ which is essential for $t \in V_f(g)$ given $S$, add $t$ to $S$. Since $V_f(g)$ is finite, the process must end. Now let us define a new value function $G$, based on the final $S$, as follows.

$$G(v) = \begin{cases} g(v) & v \notin S \\ \infty & v \in S \end{cases}$$

We claim that $G$ satisfies **A** and **C**; this contradicts **c"**. We shall use the self-explanatory notation **A**$(G)$, **C**$(G)$, **A**$(g)$ and **C**$(g)$ to avoid confusion.

We begin by proving **A**$(G)$. Let $u \in V^f(G)$. Hence also $u \in V^f(g)$. If $F(u) \cap S = \varnothing$, then **A**$(G)$ holds by **A**$(g)$. Otherwise, since $u \notin S$, no $v \in F(u) \cap S$ is essential for $u$ given $S$. Hence for every $v \in F(u) \cap S$, either $g(v) > g'(u)$ or some $w \in F(u) \setminus S$ satisfies $g(w) = g(v)$. We infer that $G'(u) = g'(u)$ and that **A**$(G)$ holds by **A**$(g)$.

We complete the proof by showing that **C**$(G)$ holds. Let $u \in V^\infty(G)$. If $G'(u) = g'(u)$ then, by construction, either $u = u_s$ or $u \in V^\infty(g)$. If $u = u_s$ then **C**$(G)$ holds since $u_s$ violates axiom **b**. Otherwise, **C**$(G)$ holds by **C**$(g)$.

Now assume $G'(u) < g'(u)$. By construction, $G'(u) = g(v)$ for some $v \in F(u) \cap S$. By **A**$(g)$, $g(v) \notin g(F(v))$. Since $G(v) = \infty(K)$ with $K \subseteq g(F(v))$, we see that $G'(u) = g(v) \notin K$. Hence **C**$(G)$ holds. □

The technical lemma is helpful because of the following lemma.

**Lemma 5** *If $g$ satisfies $\Gamma'$ then $\mathbb{P}(g) \subseteq \mathbb{P} \cup \mathbb{D}$, $\mathbb{N}(g) \subseteq \mathbb{N} \cup \mathbb{D}$ and $\mathbb{D}(g) \subseteq \mathbb{D}$.*

**Proof:** First, note that by definition and **A**, any position in $\mathbb{N}(g)$ has a successor in $\mathbb{P}(g)$; note that by **A** and **b**, all the successors of a position in $\mathbb{P}(g)$ lie in $\mathbb{N}(g)$; note that by definition, no successor of a position in $\mathbb{D}(g)$ lies in $\mathbb{P}(g)$; and note that any position in $\mathbb{D}(g)$ has a successor in $\mathbb{D}(g)$, by definition and **C**.

We claim that when starting at a position in $\mathbb{P}(g)$, a player can't win; that when starting at a position in $\mathbb{N}(g)$, an optimal player won't lose; and that when starting at a position in $\mathbb{D}(g)$, an optimal player will neither win nor lose. This will imply the lemma.

Indeed, a player starting at $\mathbb{N}(g)$ can always move to a position in $\mathbb{P}(g)$, and from there, the other player can (if at all) only move to a position in $\mathbb{N}(g)$; thus an optimal player starting at $\mathbb{N}(g)$ won't lose, and a player starting at a position in $\mathbb{P}(g)$ can't win.

Now consider a player starting at $\mathbb{D}(g)$, and let us look at the resulting game. If the game ever exits $\mathbb{D}(g)$, the player exiting $\mathbb{D}(g)$ moves to a position in $\mathbb{N}(g)$, hence that player cannot win. Thus a player starting at $\mathbb{D}(g)$ won't lose if she always goes from a position in $\mathbb{D}(g)$ to another position in $\mathbb{D}(g)$. By symmetry, a player starting at a position in $\mathbb{D}(g)$ cannot win. $\square$

A similar lemma is the following.

**Lemma 6** *If $g$ satisfies $\Gamma'$ then $\mathbb{P}_2(g) \subseteq \mathbb{P}_2 \cup \mathbb{D}_2$, $\mathbb{N}_2(g) \subseteq \mathbb{N}_2 \cup \mathbb{D}_2$ and $\mathbb{D}_2(g) \subseteq \mathbb{D}_2$.*

**Proof:** As in the proof of lemma 5, it is enough to show that (a) any position in $\mathbb{N}_2(g)$ has a successor in $\mathbb{P}_2(g)$; (b) all the successors of a position in $\mathbb{P}_2(g)$ are in $\mathbb{N}_2(g)$; (c) no successor of $\mathbb{D}_2(g)$ lies in $\mathbb{P}_2(g)$; and (d) any position in $\mathbb{D}_2(g)$ has a successor in $\mathbb{D}_2(g)$.

We begin by proving fact (a). If $(u, v) \in \mathbb{N}_2^f(g)$, the fact follows from **A**. If, on the other hand, $(u, v) \in \mathbb{N}_2^\infty(g)$, the fact follows by definition.

We move on to fact (b). Consider $(u, v) \in \mathbb{P}_2(g)$, and suppose we move to $(u, w)$, where $w \in F(v)$. If $w \in V^f(g)$, then the fact follows by definition and **A**. If, on the other hand, $w \in V^\infty(g)$, the fact follows from **b**.

We continue with fact (c). Consider $(u, v) \in \mathbb{D}_2(g)$, and suppose we move to $(u, w) \in \mathbb{P}_2(g)$, where $w \in F(v)$. Since $\mathbb{P}_2(g) \subseteq (V_2^f(g))^2$, we must have $(u, v) \in \mathbb{D}_2^f(g)$ and $g(v) = \infty(K)$. However, by definition $g(u) \notin K$, hence $g(u) \neq g(w)$, contradicting the definition of $\mathbb{P}_2(g)$.

Finally, we prove fact (d). Suppose first that $(u, v) \in \mathbb{D}_2^f(g)$, say $g(v) = \infty$. If $g(u) \neq g'(v)$, then we can move to $(w, v) \in \mathbb{D}_2^f(g)$, where $w \in F(u)$ satisfies $g(w) = g'(v)$, by definition and **A**. If, on the other hand, $g(u) = g'(v)$, we can move to $(u, w) \in \mathbb{D}_2^f(g)$, where $w \in F(v)$ satisfies $g(w) = \infty(K)$ and $g(u) = g'(v) \notin K$, by **C**.

Suppose otherwise that $(u, v) \in \mathbb{D}_2^\infty(g)$. Then we can move to $(u, w) \in \mathbb{D}_2^\infty(g)$, where $w \in V^\infty(g)$, by **C**. $\square$

**Corollary 7** *If $g$ satisfies $\Gamma'$ then $\mathbb{N}^\infty(g) \subseteq \mathbb{N}^\infty \cup \mathbb{D}$.*

**Proof:** Let $v \in \mathbb{N}^\infty(g)$, and suppose that $v \in \mathbb{N}$. If $v \in \mathbb{N}^f$ then $(v, v) \in \mathbb{P}_2$, whereas if $v \in \mathbb{N}^\infty$ then $(v, v) \in \mathbb{D}_2$. Since $(v, v) \in \mathbb{D}_2(g) \subseteq \mathbb{D}_2$, we see that $v \in \mathbb{N}^\infty$. $\square$

Lemmas 4 and 5 and corollary 7 together imply theorem 1.