Why is the number of 123- and 132-avoiding permutations equal to the number of binary trees?

1 What are restricted permutations?

We shall deal with permutations avoiding some specific patterns. For us, a permutation on $[1, n]$ will mean a list of the integers 1 up to n in some order. For example, here are all permutations on $[1, 3]$:

$$123, \quad 132, \quad 213, \quad 231, \quad 312, \quad 321.$$

The corresponding (normal) permutation is obtained by letting the element i go into the element in position i. For example, in 1432, 1 goes to 1, 2 goes to 4, 3 goes to 3 and 4 goes to 2, hence the permutation is (24).

The term *pattern* will mean for us the same thing as *permutation*, only with different semantics. We say that a permutation σ *avoids* a pattern τ if no *subperm* of σ is ordered as the pattern τ. By a *subperm* we mean a certain number of elements taken *in order* but not necessarily *together*. For example, the seven non-empty subperms of 123 are

$$1, \quad 2, \quad 3, \quad 12, \quad 13, \quad 23, \quad 123.$$

A subperm is said to be *ordered according to* some pattern of the same length n if for any $1 \leq k \leq n$, the k-largest element is located in the same position in both the subperm and the pattern. For example, the subperm $\alpha\beta\gamma$ is ordered as the pattern 132 if $\alpha < \gamma < \beta$.

2 Wilf classes

Let $A_n(\tau)$ denote the number of permutation on $[1, n]$ avoiding the pattern τ. For example, $A_3(123) = 5$, for obvious reasons. For each pattern length m, we say that the patterns τ and τ' are in the same Wilf class if $A_n(\tau) = A_n(\tau')$ for any permutation length n. There are three simple operations on patterns which are guaranteed to preserve the Wilf class: reversing, complementing and inverting.

2.1 Reversing

The *reverse* of a pattern τ, denoted τ^R, is just the pattern τ reversed as a text string. For example, $123^R = 321$. It is clear that σ avoids τ if, and only if, σ^R avoids τ^R; it is further clear that reversing is a permutation on S_n, hence τ and τ^R share the same Wilf class.

2.2 Complementing

The *complement* of a pattern τ, denoted τ^C, is obtained by subtracting each element of τ from $m + 1$, m being the length of τ. For example, to complement 1432 we subtract each element from 5 to obtain $1432^C = 4123$. It is easy to see that σ avoids τ if, and only if, σ^C avoids τ^C, since complementing corresponds to switching $<$ and $>$; it is further clear that complementing is a permutation on S_n, hence τ and τ^C share the same Wilf class.

2.3 Inverting

The *inverse* of a pattern τ, denoted τ^{-1}, is obtained by inverting τ as a permutation. Alternatively, one encodes τ using *positional notation*: that is, the first element is the position of 1 in τ, the second is the position of 2, and so on. For example, consider inverting the permutation 1423. According to the first definition, $1423^{-1} = (243)^{-1} = (234) = 1342$. According to the second definition, the position of 1 is 1, that of 2 is 3, that of 3 is 4, that of 4 is 2. Thus we also get 1342.

Why are the two definitions equivalent? Let us look at a permutation in a third way: as a set of (i, j) pairs, meaning i goes to j. For example, $1423 = \{(1, 1), (2, 4), (3, 2), (4, 3)\}$. Inverting the permutation means reversing the pairs, which amounts to the positional notation.

Now we can easily show that a permutation σ avoids the pattern τ if, and only if, σ^{-1} avoids τ^{-1}: for this amounts only to decoding the pairs in the set representation the opposite way. Hence τ and τ^{-1} belong to the same Wilf class.

3 Patterns of length 3

Now we set out to find the partition into Wilf classes of the patterns of length 3. The six possible patterns were already listed, and we can remark that

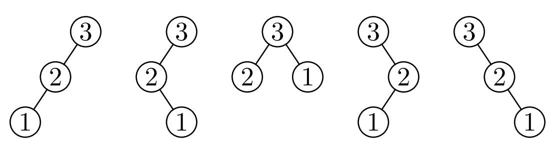
$$123^R = 321, \quad 132^C = 312, \quad 312^{-1} = 231, \quad 312^R = 213.$$

Hence, a-priori there are at most two Wilf classes. We will show bellow that actually $A_n(123) = A_n(132) = C_n$, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the nth Catalan number, hence there is exactly one Wilf class. This will be done by providing an explicit bijection between permutations avoiding a specific pattern and rooted binary trees on n nodes. This will prove the result, since C_n is the number of such trees. Note that when we count this number, we distinguish between a single left son and a single right son (thus there are five trees having three vertices).

3.1 132-avoiding permutations

The connection between binary trees and 132-avoiding permutations is quite straightforward. What does it mean for a permutation σ to avoid 132? Suppose $|\sigma| = n$ (that is, σ is of size n). Let us write $\sigma = \alpha n \beta$. Then clearly α and β must be 132-avoiding, and furthermore, every element in α must be larger than any element in β. It is also clear that if these two conditions hold, then σ avoids 132.

We can now outline a conversion procedure from 132-avoiding permutations into binary trees. To convert $\sigma = \alpha n \beta$ into a tree, we first convert α and β into trees (we have to subtract $|\beta|$ from α first). Then we create a new rooted tree, having the converted α as left branch, and the converted β as right branch. For example, here are all 132-avoiding permutation of length 3 converted into binary trees, with self-explanatory numbering:



123 213 231 312 321

We can easily prove inductively that different permutation give rise to different trees: if the position of n in the two permutations is different, then the number of nodes in each branch from the root will clearly be different; otherwise, things come down to comparing the corresponding left branches and right branches; at least one of these must be different by induction.

Conversely, to convert a binary tree into a 132-avoiding permutation, we employ the following mechanism. We convert recursively the left branch into a permutation α, and the right branch into a permutation β; an empty branch translates into an empty permutation. Then we add $|\beta|$ to α, getting γ, and finally we get the converted permutation $\gamma n \beta$. Thus we have shown that $A_n(132) = C_n$.

3.2 123-avoiding permutations

The connection between binary trees and 123-avoiding permutations is more obscure and less natural. We must first study the recursive structure of 123-avoiding permutations, as we have done in the previous case.

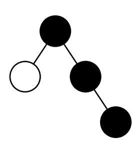
We shall classify 123-avoiding permutations according to the length of the (largest) *falling sequence* starting the permutation σ, denoted $\ell(\sigma)$. For example, the permutation 3214 starts with a descending run of length three: $\underline{321}4$. There is a strong connection between this length and the location of the highest element n, which will be very helpful.

The connection is that n always begins or (immediately) follows the falling sequence. For if we are given a 123-avoiding permutation σ which does not start with n, let us write $\sigma = \alpha\beta n\gamma$, where α is the falling sequence. The falling sequence is always of length at least one. If β was not empty, then take the last element a of α, and the first element b of β. Since clearly $a < b < n$, we have found an occurrence of the forbidden pattern 123 in σ — and that cannot be. Hence n must follow the falling sequence.

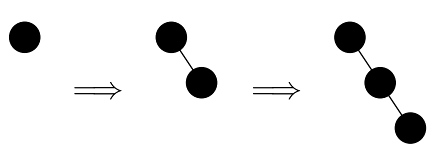
Now consider all 123-avoiding permutations σ on $[1, n]$ with $\ell(\sigma) = i$. What happens if we extract n? There are two possibilities: either $\sigma = n\alpha\beta$ or $\sigma = \alpha n\beta$, where $n\alpha$ or α is the falling sequence. Denoting the permutation obtained by removing n as σ', we get in both cases $\sigma' = \alpha\beta$; however, in the first case, α remains the falling sequence, hence $\ell(\sigma') = i-1$, whereas in the second case, we only know that $\ell(\sigma') \geq i$.

Conversely, we can make any permutation σ' on $[1, n-1]$ with $\ell(\sigma') \geq i-1$ a new permutation σ on $[1, n]$ with $\ell(\sigma') = i$ by inserting n at the proper place: at the beginning if $\ell(\sigma') = i-1$, or after exactly i elements if $\ell(\sigma') \geq i$.

What does all this have to do with binary trees? There is a binary-tree equivalent to a falling sequence: a *right-branching*. The *right-branching* of a (rooted) binary tree is defined as the nodes on the path beginning with the root and always branching to the right, or sometimes their number (naturally we could also have used the analog concept of left-branching). For example, the following tree has a right branching of length three, which is highlighted:

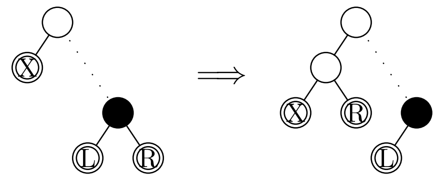


As we have seen, permutations σ with $\ell(\sigma) = i$ can be reduced to permutations σ' with $\ell(\sigma') \geq i-1$ with one element less. We can use this in order to transform 123-avoiding permutations to binary trees, where our goal is to generate a tree with a right-branching equal to $\ell(\sigma)$. Given a permutation σ, we discern among two cases: the first is when n starts the permutation, the second when it does not. The first case is easy: we drop the n, getting a permutation σ' with $\ell(\sigma') = \ell(\sigma) - 1$. We generate its corresponding tree recursively, then append it as the right son of a new root to get the tree for σ, which has the appropriate right branching. We demonstrate the process, highlighting the right-branching:

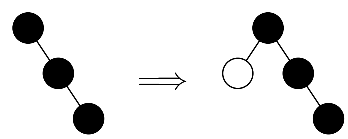


1 21 321

The second case is a bit more involved. We need to cut the right-branching in order to make it of length exactly $\ell(\sigma)$. All that we know is that $\ell(\sigma') \geq \ell(\sigma)$, and so this cutting can always be done. To explain how this cutting is done, let us draw a diagram:

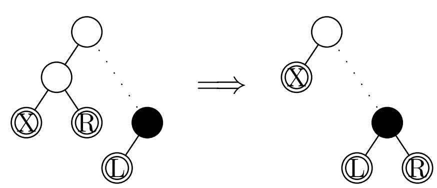


Note that doubled nodes signify subtrees, which might turn out to be empty. The cut should be made after $i - 1$ right branches — that is, there should be i vertices from the root to the full vertex, including both. As an example, we complete the construction of the tree corresponding to 3214:



We can easily prove inductively that different permutations give rise to different trees: evidently, permutations of two different types (that is, our cases) give rise to different trees, one where the root has a left son, one when it hasn't. Two permutations of the first type clearly lead to different trees, and for permutations of the second type, we consider two cases: in the first case, the positions of n are different — this clearly leads to different trees (with different right-branchings); in the second case, n appears in the same position, and now we have to resort to induction in order to see that we get two different trees.

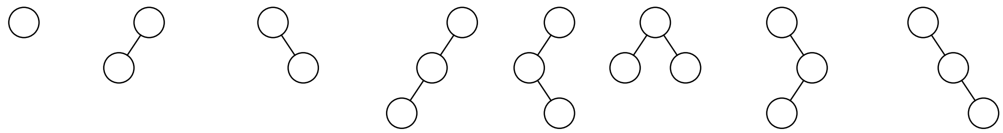
To deconstruct a tree into a permutation, we reverse the construction: first we remember the right-branching ℓ, where we put $\ell = 0$ if the root has no left son (for reasons soon to be explained). Then we reduce the tree, according to two cases: if there is no left son, we just delete the root; if there is a left son, then we decompose the tree as follows:



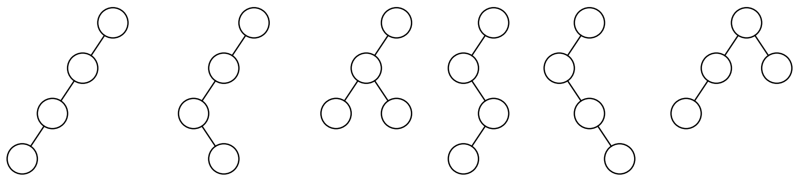
After decomposing the tree (in any of the two cases), we obtain a permutation recursively. Now we insert n after ℓ elements (that is, n will be in position $\ell + 1$). Hence we have shown that $A_n(123) = C_n$.

4 Appendix: some transforms

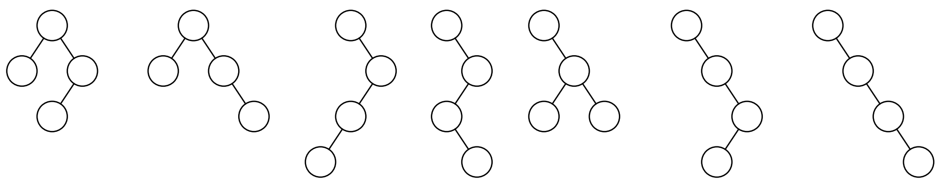
Here are some trees and their transforms; 132-transforms and bellow them 123-transforms.



1	12	21	123	213	231	312	321
1	12	21	132	231	213	312	321



1234	2134	2314	3124	3214	2341	3241
1432	2431	2413	3412	3421	2143	3241



3412	3421	4123	4213	4231	4312	4321
3142	3214	4132	4231	4213	4312	4321