Murali's basis for the slice

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Let us present Murali's construction of the GZ basis for the slice, and show how it results in the explicit formula given in the first author's work.

We will construct, for each $n \geq 1$ and each $0 \leq d \leq n/2$, an orthogonal basis $\mathcal{B}_{n,d}$ of homogeneous degree d functions. The orthogonal basis for the slice $\binom{[n]}{d}$ is the union of $\mathcal{B}_{n,e}$ over $0 \leq e \leq \min(d, n-d)$. (We later comment on how this corresponds to Murali's description, which is a bit different.)

For now, we just present the construction, not proving any of its properties. In brief, one inductively proves that the elements are orthogonal, perhaps using their harmonicity (being annihilated by the Down operator, the adjoint of the Up operator described below). One also explicitly computes their norms to show that they don't vanish. That they form a basis follows from a dimension argument. Finally, to show that they are the GZ basis, one shows that they are eigenfunctions of the YJM elements, explicitly computing the eigenvalues along the way. We skip all that for now.

We can identify each function in $\mathcal{B}_{n,d}$ with the corresponding subset of $\binom{[n]}{d}$ using the following observation: the monomial $x_{i_1} \cdots x_{i_d}$ (where all $i_1, \ldots, i_d$ are distinct) is the same as the delta function of $\{i_1, \ldots, i_d\}$.

The construction will use the Up operator U from $\binom{[n]}{d}$ to $\binom{[n]}{d+1}$, which maps each element in $\binom{[n]}{d}$ to all subsets of $\binom{[n]}{d+1}$ containing it (and extended linearly).

We can now explain how Murali thinks of his basis. Each element of $\mathcal{B}_{n,d}$ is a basis element over $\binom{[n]}{d}$. To get the corresponding basis elements for the slices $\binom{[n]}{d+1}, \ldots, \binom{[n]}{n-d}$, Murali repeatedly applies the Up operator. The Up operator takes a degree d monomial to a degree d monomial, multiplied by some factor depending on d and on the slice you start with. Since all elements of $\mathcal{B}_{n,d}$ are homogeneous, the effect is to multiply the homogeneous representation by some constant depending on d and on the slice $\binom{[n]}{e}$ (where $d \leq e \leq n-d$). The range of e is chosen to guarantee that the constant is non-zero.

Let us now describe the construction. The starting point is $\mathcal{B}_{1,0} = \{1\}$. Each function in $\mathcal{B}_{n-1,d}$ also appears in $\mathcal{B}_{n,d}$ (this is "0-lifting"). If $d < (n-1)/2$, then we also get a new function:

$$f \mapsto \mathsf{X}(f) := (n-2d-1) f x_n - Uf.$$

Here multiplying by x_n corresponds to "1-lifting", and U is applied on the original $n-1$ coordinates.

We will show that each member of $\mathcal{B}_{n,d}$ is always of the following form, for $j_1 < \cdots < j_d$:

$$B(i_1, \ldots, i_d) = \sum_{\substack{j_1, \ldots, j_d : \\ j_t < i_t \\ i_s \neq j_t}} (x_{i_1} - x_{j_1}) \cdots (x_{i_d} - x_{j_d}).$$

In particular, we will show that

$$\mathsf{X}(B(i_1, \ldots, i_d)) = B(i_1, \ldots, i_d, n).$$

We only expand basis elements if there is "space", that is, if the basis element doesn't vanish. This is the case for a basis element $B(k_1, \ldots, k_d)$ if for each $1 \leq e \leq d$, we have $|[k_e] \setminus \{k_1, \ldots, k_e\}| \geq e$. In our expansion procedure, we are guaranteed that this condition will hold for all proper prefixes, and the fact that we only expand if $d < (n-1)/2$ guarantees that it holds for the entire sequence.

Consider any degree d monomial $x_{k_1} \cdots x_{k_d}$. The effect of the Up operator is to multiply it by the sum of all elements not appearing in it, that is, $x_1 + \cdots + x_{n-1} - x_{k_1} - \cdots - x_{k_d}$. This is because we're applying the Up operator from $\binom{[n]}{d}$ to $\binom{[n]}{d+1}$; in both cases, we can identify monomials with points.

Consider now any element in the sum defining $B(j_1, \ldots, j_d)$, for example

$$(x_{i_1} - x_{j_1}) \cdots (x_{i_d} - x_{j_d}).$$

What happens when we apply the Up operator? Let us start with the case $d = 1$ for concreteness: (hats represent elements removed from the sum)

$$U(x_{i_1} - x_{j_1}) = x_{i_1}(x_1 + \cdots + \widehat{x_{i_1}} + \cdots + x_{n-1}) - x_{j_1}(x_1 + \cdots + \widehat{x_{j_1}} + \cdots + x_{n-1}).$$

Notice that both terms contain $x_{i_1} x_{j_1}$, which we can cancel, obtaining

$$U(x_{i_1} - x_{j_1}) = (x_{i_1} - x_{j_1})(x_1 + \cdots + \widehat{x_{j_1}} + \cdots + \widehat{x_{i_1}} + \cdots + x_{n-1}).$$

A similar phenomenon happens in general due to the alternating signs, and we get

$$U[(x_{i_1} - x_{j_1}) \cdots (x_{i_d} - x_{j_d})] = (x_{i_1} - x_{j_1}) \cdots (x_{i_d} - x_{j_d}) \sum_{\substack{1 \le k \le n-1 \\ k \neq i_t, j_t}} x_k.$$

Note that there are $n - 2d - 1$ elements in the sum.

Putting everything together, we get:

$$\begin{aligned}
\mathsf{X}(B(i_1, \ldots, i_d)) &= \sum_{\substack{j_1, \ldots, j_d : \\ j_t < i_t \\ i_s \neq j_t}} (x_{i_1} - x_{j_1}) \cdots (x_{i_d} - x_{j_d}) \left[(n - 2d - 1)x_n - \sum_{\substack{1 \le k \le n-1 \\ k \neq i_t, j_t}} x_k \right] \\
&= \sum_{\substack{j_1, \ldots, j_d : \\ j_t < i_t \\ i_s \neq j_t}} (x_{i_1} - x_{j_1}) \cdots (x_{i_d} - x_{j_d}) \sum_{\substack{1 \le k \le n-1 \\ k \neq i_t, j_t}} (x_n - x_k) \\
&= B(i_1, \ldots, i_d, n).
\end{aligned}$$

1 Gram–Schmidt Orthogonalization

Consider some degree d. We arrange the basis elements in lexicographical order with respect to $x_n > x_{n-1} > \cdots > x_1$. For example,

$$B(1,2) < B(1,3) < B(2,3) < B(1,4) < B(2,4) < B(3,4) < \cdots$$

Under this order, the basis B is a Gröbner basis for the harmonic multilinear degree d polynomials.

For $i_1, \ldots, i_d$, define

$$C(i_1, \ldots, i_d) = \{(x_{i_1} - x_{j_1}) \cdots (x_{i_d} - x_{j_d}) : j_t < i_t,\ i_t \neq j_s\}.$$

We make three observations. First, every element in $C(i_1, \ldots, i_d)$ is in the span of $B(i'_1, \ldots, i'_d)$ for $(i'_1, \ldots, i'_d) \le (i_1, \ldots, i_d)$. This follows from the Gröbner basis property. Second, the difference of any two elements is in the span of $B(i'_1, \ldots, i'_d)$ for $(i'_1, \ldots, i'_d) < (i_1, \ldots, i_d)$. Indeed, it suffices to consider two elements differing in, say, j_t. If the two values are $j'_t < j''_t$, then the difference is in $C(i_1, \ldots, j''_t, \ldots, i_d)$, and so in the required span by the first observation. Third, $B(i_1, \ldots, i_d)$ is the sum of elements in $C(i_1, \ldots, i_d)$.

Now suppose that we choose one element $f_{i_1, \ldots, i_d} \in C(i_1, \ldots, i_d)$ for each top set $i_1, \ldots, i_d$, and arrange them in increasing order of $(i_1, \ldots, i_d)$. I claim that Gram–Schmidt will produce the basis $B(i_1, \ldots, i_d)$ (up to scalar multiples). The proof is by induction. It suffices to show that $B(i_1, \ldots, i_d)$ is in the span of $f_{i_1, \ldots, i_d}$ and all preceding B's. Indeed, by the second observation, all elements in $C(i_1, \ldots, i_d)$ are in this span. The third observation then completes the proof.