ANOTHER PROOF OF CAUCHY'S GROUP THEOREM[1]

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Since $ab = 1$ implies $ba = b(ab)b^{-1} = 1$, the identities are symmetrically placed in the group table of a finite group. Each row of a group table contains exactly one identity and thus if the group has even order, there are an even number of identities on the main diagonal. Therefore, $x^2 = 1$ has an even number of solutions.

Generalizing this observation, we obtain a simple proof of Cauchy's theorem. For another proof see [**1**].

CAUCHY'S THEOREM. *If the prime p divides the order of a finite group G, then G has kp solutions to the equation $x^p = 1$.*

Let G have order n and denote the identity of G by 1. The set

$$S = \{(a_1, \cdots, a_p) \mid a_i \in G,\, a_1 a_2 \cdots a_p = 1\}$$

has n^{p-1} members. Define an equivalence relation on S by saying two p-tuples are equivalent if one is a cyclic permutation of the other.

If all components of a p-tuple are equal then its equivalence class contains only one member. Otherwise, if two components of a p-tuple are distinct, there are p members in the equivalence class.

Let r denote the number of solutions to the equation $x^p = 1$. Then r equals the number of equivalence classes with only one member. Let s denote the number of equivalence classes with p members. Then $r + sp = n^{p-1}$ and thus $p \mid r$.

Reference

1. G. A. Miller, On an extension of Sylow's theorem, Bull. Amer. Math. Soc., vol. 4, 1898, pp. 323–327.

[1] American Mathematical Monthly, Vol. 66 (February 1959), p. 119.