Some fast matrix multiplication identities

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1 Warmup identity ($\omega \approx 2.404$)

Identity:

$$\epsilon^3 \sum_{i=1}^{q} \left(x_0^{[0]} y_i^{[1]} z_i^{[1]} + x_i^{[1]} y_0^{[0]} z_i^{[1]} + x_i^{[1]} y_i^{[1]} z_0^{[0]} \right) + O(\epsilon^4) =$$
$$\epsilon \sum_{i=1}^{q} (x_0^{[0]} + \epsilon x_i^{[1]})(y_0^{[0]} + \epsilon y_i^{[1]})(z_0^{[0]} + \epsilon z_i^{[1]}) -$$
$$\left(x_0^{[0]} + \epsilon^2 \sum_{i=1}^{q} x_i^{[1]} \right) \left(y_0^{[0]} + \epsilon^2 \sum_{i=1}^{q} y_i^{[1]} \right) \left(z_0^{[0]} + \epsilon^2 \sum_{i=1}^{q} z_i^{[1]} \right) +$$
$$(1 - q\epsilon) x_0^{[0]} y_0^{[0]} z_0^{[0]}.$$

Interpretation:

$$\underline{\mathrm{R}}(\langle 1, 1, q\rangle^{0,1,1} + \langle q, 1, 1\rangle^{1,0,1} + \langle 1, q, 1\rangle^{1,1,0}) \leq q + 2.$$

2 Main identity ($\omega \approx 2.387$)

Identity:

$$\epsilon^3 \left[\sum_{i=1}^{q} \left(x_0^{[0]} y_i^{[1]} z_i^{[1]} + x_i^{[1]} y_0^{[0]} z_i^{[1]} + x_i^{[1]} y_i^{[1]} z_0^{[0]} \right) + x_0^{[0]} y_0^{[0]} z_{q+1}^{[2]} + x_0^{[0]} y_{q+1}^{[2]} z_0^{[0]} + x_{q+1}^{[2]} y_0^{[0]} z_0^{[0]} \right] + O(\epsilon^4) =$$
$$\epsilon \sum_{i=1}^{q} (x_0^{[0]} + \epsilon x_i^{[1]})(y_0^{[0]} + \epsilon y_i^{[1]})(z_0^{[0]} + \epsilon z_i^{[1]}) -$$
$$\left(x_0^{[0]} + \epsilon^2 \sum_{i=1}^{q} x_i^{[1]} \right) \left(y_0^{[0]} + \epsilon^2 \sum_{i=1}^{q} y_i^{[1]} \right) \left(z_0^{[0]} + \epsilon^2 \sum_{i=1}^{q} z_i^{[1]} \right) +$$
$$(1 - q\epsilon)(x_0^{[0]} + \epsilon^3 x_{q+1}^{[2]})(y_0^{[0]} + \epsilon^3 y_{q+1}^{[2]})(z_0^{[0]} + \epsilon^3 z_{q+1}^{[2]}).$$

Interpretation:

$$\underline{\mathrm{R}}(\langle 1, 1, q\rangle^{0,1,1} + \langle q, 1, 1\rangle^{1,0,1} + \langle 1, q, 1\rangle^{1,1,0} + \langle 1, 1, 1\rangle^{0,0,2} + \langle 1, 1, 1\rangle^{0,2,0} + \langle 1, 1, 1\rangle^{2,0,0}) \leq q + 2.$$

3 Identity squared ($\omega \approx 2.375$)

Identity squared (left-hand side):

(a) 3 terms similar to $\langle 1,1,1\rangle^{0,0,4}$, coming from

$$\langle 1,1,1\rangle^{0,0,2} \otimes \langle 1,1,1\rangle^{0,0,2}.$$

(b) 6 terms similar to $\langle 1,1,2q\rangle^{0,1,3}$, coming from

$$\langle 1,1,q\rangle^{0,1,1} \otimes \langle 1,1,1\rangle^{0,0,2} + \langle 1,1,1\rangle^{0,0,2} \otimes \langle 1,1,q\rangle^{0,1,1}.$$

(c) 3 terms similar to $\langle 1,1,q^2+2\rangle^{0,2,2}$, coming from

$$\langle 1,1,1\rangle^{0,2,0} \otimes \langle 1,1,1\rangle^{0,0,2} + \langle 1,1,1\rangle^{0,0,2} \otimes \langle 1,1,1\rangle^{0,2,0} + \langle 1,1,q\rangle^{0,1,1} \otimes \langle 1,1,q\rangle^{0,1,1}.$$

(d) 3 terms similar to T_4^{112}, coming from

$$\langle 1,q,1\rangle^{1,1,0} \otimes \langle 1,1,1\rangle^{0,0,2} + \langle 1,1,1\rangle^{0,0,2} \otimes \langle 1,q,1\rangle^{1,1,0} + \langle q,1,1\rangle^{1,0,1} \otimes \langle 1,1,q\rangle^{0,1,1} + \langle 1,1,q\rangle^{0,1,1} \otimes \langle q,1,1\rangle^{1,0,1}.$$

The tensor T_4 is given by

$$T_4 = \langle 1,q,1\rangle^{10,10,02} + \langle 1,q,1\rangle^{01,01,20} + \langle q,1,q\rangle^{10,01,11} + \langle q,1,q\rangle^{01,10,11}.$$

Interpretation:

$$\underline{\mathrm{R}}(3\langle 1,1,1\rangle^{0,0,4} + 6\langle 1,1,2q\rangle^{0,1,3} + 3\langle 1,1,q^2+2\rangle^{0,2,2} + 3T_4^{1,1,2}) \le (q+2)^2.$$

4 Strassen's identity ($\omega \approx 2.48$)

Identity:

$$\epsilon \sum_{i=1}^{q} (x_i^{[1]} y_0^{[0]} z_i^{[1]} + x_0^{[0]} y_i^{[1]} z_i^{[1]}) + O(\epsilon^2) =$$
$$\sum_{i=1}^{q} (x_0^{[0]} + \epsilon x_i^{[1]})(y_0^{[0]} + \epsilon y_i^{[1]}) z_i - x_0^{[0]} y_0^{[0]} \sum_{i=1}^{q} z_i.$$

Interpretation:

$$\underline{\mathrm{R}}(\langle q,1,1\rangle^{1,0,1} + \langle 1,1,q\rangle^{0,1,1}) \le q+1.$$

5 For the record

Following table is taken from François Le Gall, *Powers of tensors and fast matrix multiplication*, arXiv:1401.7714.

Who	Power	Bound
C.-W.	1	$\omega < 2.3871900$
C.-W.	2	$\omega < 2.3754770$
Stothers	4	$\omega < 2.3729269$
V.-Williams	8	$\omega < 2.3728642$
Le Gall	16	$\omega < 2.3728640$
Le Gall	32	$\omega < 2.3728639$