Some fast matrix multiplication identities

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1 Strassen's algorithm ($\omega \approx 2.81$)

Original algorithm:

$$
\begin{aligned}
m_1 &= (a_{11} + a_{22})(b_{11} + b_{22}) \\
m_2 &= (a_{21} + a_{22})b_{11} \\
m_3 &= a_{11}(b_{12} - b_{22}) \\
m_4 &= a_{22}(b_{21} - b_{11}) \\
m_5 &= (a_{11} + a_{12})b_{22} \\
m_6 &= (a_{21} - a_{11})(b_{11} + b_{12}) \\
m_7 &= (a_{12} - a_{22})(b_{21} + b_{22}) \\
c_{11} &= m_1 + m_4 - m_5 + m_7 \\
c_{12} &= m_3 + m_5 \\
c_{21} &= m_2 + m_4 \\
c_{22} &= m_1 - m_2 + m_3 + m_6
\end{aligned}
$$

Tensor notation:

$$
\begin{aligned}
&\sum_{i,j,k=1}^{2} a_{ik}b_{kj}c_{ij} = \\
&(a_{11} + a_{22})(b_{11} + b_{22})(c_{11} + c_{22}) + \\
&(a_{21} + a_{22})b_{11}(c_{21} - c_{22}) + \\
&a_{11}(b_{12} - b_{22})(c_{12} + c_{22}) + \\
&a_{22}(b_{21} - b_{11})(c_{11} + c_{21}) + \\
&(a_{11} + a_{12})b_{22}(-c_{11} + c_{12}) + \\
&(a_{21} - a_{11})(b_{11} + b_{12})c_{22} + \\
&(a_{12} - a_{22})(b_{21} + b_{22})c_{11}.
\end{aligned}
$$

Interpretation: $R(\langle 2, 2, 2\rangle) \leq 7$.

2 Bini's identity ($\omega \approx 2.78$)

Identity:

$$
\begin{aligned}
&\epsilon(a_{11}b_{11}c_{11} + a_{11}b_{12}c_{21} + a_{12}b_{21}c_{11} + a_{12}b_{22}c_{21} + a_{21}b_{11}c_{12} + a_{21}b_{12}c_{22}) + \\
&\epsilon^2(a_{11}b_{22}c_{21} + a_{11}b_{11}c_{12} + a_{12}b_{21}c_{22} + a_{21}b_{21}c_{22}) = \\
&(a_{12} + \epsilon a_{11})(b_{12} + \epsilon b_{22})c_{21} + \\
&(a_{21} + \epsilon a_{11})b_{11}(c_{11} + \epsilon c_{12}) - \\
&a_{12}b_{12}(c_{11} + c_{21} + \epsilon c_{22}) - \\
&a_{21}(b_{11} + b_{12} + \epsilon b_{21})c_{11} + \\
&(a_{12} + a_{21})(b_{12} + \epsilon b_{21})(c_{11} + \epsilon c_{22}).
\end{aligned}
$$

Interpretation:

$$
\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}.
$$

Putting two copies together:

$$
\underline{\mathrm{R}}(\langle 3, 2, 2\rangle) \leq 10.
$$

3 Schönhage's identities ($\omega \approx 2.55$, $\alpha \approx 0.1402$)

Identity:

$$
\begin{aligned}
&\epsilon^2 \left(\sum_{i=1}^{4} \sum_{j=1}^{4} A_i B_j C_{ji} + \sum_{i=1}^{3} \sum_{j=1}^{3} X_{3i+j} Y_{3i+j} Z \right) + O(\epsilon^3) = \\
&\sum_{i=1}^{3} \sum_{j=1}^{3} (A_i + \epsilon X_{3i+j})(B_j + \epsilon Y_{3i+j})(\epsilon^2 C_{ji} + Z) + \\
&\sum_{i=1}^{3} A_i (B_4 - \epsilon \sum_{j=1}^{3} Y_{3i+j})(\epsilon^2 C_{4i} + Z) + \sum_{j=1}^{3} (A_4 - \epsilon \sum_{i=1}^{3} X_{3i+j}) B_j (\epsilon^2 C_{j4} + Z) + \\
&A_4 B_4 (\epsilon^2 C_{44} + Z) - \left(\sum_{i=1}^{4} A_i \right) \left(\sum_{j=1}^{4} B_j \right) Z.
\end{aligned}
$$

Interpretation:

$$\underline{\mathrm{R}}(\langle 4,1,4 \rangle \oplus \langle 1,9,1 \rangle) \le 17.$$

Similarly,

$$\underline{\mathrm{R}}(\langle 3,1,3 \rangle \oplus \langle 1,4,1 \rangle) \le 10.$$

4 Another identity by Schönhage ($\alpha \approx 0.17227$)

Identity:

$$
\begin{aligned}
&\epsilon^2 (a_{11} b_{11} c_{11} + a_{11} b_{12} c_{21} + a_{12} b_{21} c_{11} + a_{13} b_{31} c_{11} + a_{22} b_{21} c_{12} + a_{23} b_{31} c_{12}) + O(\epsilon^3) = \\
&(a_{11} + \epsilon^2 a_{12})(b_{21} + \epsilon^2 b_{11}) c_{11} + \\
&(a_{11} + \epsilon^2 a_{13}) b_{31} (c_{11} - \epsilon c_{21}) + \\
&(a_{11} + \epsilon^2 a_{22})(b_{21} - \epsilon b_{12}) c_{12} + \\
&(a_{11} + \epsilon^2 a_{23})(b_{31} + \epsilon b_{12})(c_{12} + \epsilon c_{21}) - \\
&a_{11} (b_{21} + b_{31})(c_{11} + c_{12}).
\end{aligned}
$$

Interpretation:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & 0 \\ b_{31} & 0 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}.$$

Border rank of this partial matrix multiplication is 5.