The number e is transcendental[1].

Proof. In the proof we shall use the standard notation $f^{(i)}(x)$ to denote the ith derivative of $f(x)$ with respect to x.

Suppose that $f(x)$ is a polynomial of degree r with real coefficients. Let $F(x) = f(x)+f^{(1)}(x)+f^{(2)}(x)+\cdots+f^{(r)}(x)$. We compute $(d/dx)(e^{-x}F(x))$; using the fact that $f^{(r+1)}(x) = 0$ (since $f(x)$ is of degree r) and the basic property of e, namely that $(d/dx)e^x = e^x$, we obtain $(d/dx)(e^{-x}F(x)) = -e^{-x}f(x)$.

The mean value theorem asserts that if $g(x)$ is a continuously differentiable, single-valued function on the closed interval $[x_1, x_2]$ then

$$\frac{g(x_1) - g(x_2)}{x_1 - x_2} = g^{(1)}(x_1 + \theta(x_2 - x_1)), \quad \text{where} \quad 0 < \theta < 1.$$

We apply this to our function $e^{-x}F(x)$ which certainly satisfies all the required conditions for the mean value theorem on the closed interval $[x_1, x_2]$ where $x_1 = 0$ and $x_2 = k$, where k is any positive integer. We then obtain that $e^{-k}F(k) - F(0) = -e^{-\theta_k k}f(\theta_k k)k$, where θ_k depends on k and is some real number between 0 and 1. Multiplying this relation through by e^k yields $F(k) - F(0)e^k = -e^{(1-\theta k)k}f(\theta_k k)$. We write this out explicitly:

$$(1) \qquad \begin{aligned} F(1) - eF(0) &= -e^{(1-\theta_1)}f(\theta_1) = \epsilon_1 \\ F(2) - e^2F(0) &= -2e^{2(1-\theta_2)}f(2\theta_2) = \epsilon_2 \\ &\vdots \\ F(n) - e^nF(0) &= -ne^{n(1-\theta_n)}f(n\theta_n) = \epsilon_n. \end{aligned}$$

Suppose now that e is an algebraic number; then it satisfies some relation of the form

$$(2) \qquad c_ne^n + c_{n-1}e^{n-1} + \cdots + c_1e + c_0 = 0,$$

where $c_0, c_1, \ldots, c_n$ are integers and where $c_0 > 0$.

In the relations (1) let us multiply the first equation by c_1, the second by c_2, and so on; adding these up we get $c_1F(1)+c_2F(2)+\cdots+c_nF(n)-F(0)(c_1e+c_2e^2+\cdots+c_ne^n) = c_1\epsilon_1+c_2\epsilon_2+\cdots+c_n\epsilon_n$.

In view of relation (2), $c_1e + c_2e^2 + \cdots + c_ne^n = -c_0$, whence the above equation simplifies to

$$(3) \qquad c_0F(0) + c_1F(1) + \cdots + c_nF(n) = c_1\epsilon_1 + \cdots + c_n\epsilon_n.$$

All this discussion has held for the $F(x)$ constructed from an arbitrary polynomial $f(x)$. We now see what all this implies for a very specific polynomial, one first used by Hermite, namely,

$$f(x) = \frac{1}{(p-1)!}x^{p-1}(1-x)^p(2-x)^p\cdots(n-x)^p.$$

Here p can be any prime number chosen so that $p > n$ and $p > c_0$. For this polynomial we shall take a very close look at $F(0), F(1), \ldots, F(n)$ and we shall carry out an estimate on the size of $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$.

When expanded, $f(x)$ is a polynomial of the form

$$\frac{(n!)^p}{(p-1)!}x^{p-1} + \frac{a_0x^p}{(p-1)!} + \frac{a_1x^{p+1}}{(p-1)!} + \cdots,$$

where $a_0, a_1, \ldots,$ are integers.

[1]Herstein, Topics in Algebra, p. 176–178.

When $i \geq p$ we claim that $f^{(i)}(x)$ is a polynomial, with coefficients which are integers all of which are multiples of p. (Prove!) *Thus for any integer j, $f^{(i)}(j)$, for $i \geq p$ is an integer and is a multiple of p.*

Now, from its very definition, $f(x)$ has a root of multiplicity p at $x = 1, 2, \ldots, n$. Thus for $j = 1, 2, \ldots, n$, $f(j) = 0$, $f^{(1)}(j) = 0, \ldots, f^{(p-1)}(j) = 0$. However, $F(j) = f(j) + f^{(1)}(j) + \cdots + f^{(p-1)}(j) + f^{(p)}(j) + \cdots + f^{(r)}(j)$; by the discussion above, for $j = 1, 2, \ldots, n$, *$F(j)$ is an integer and is a multiple of p.*

What about $F(0)$? Since $f(x)$ has a root of multiplicity $p-1$ at $x = 0$, $f(0) = f^{(1)}(0) = \cdots = f^{(p-2)}(0) = 0$. For $i \geq p$, $f^{(i)}(0)$ is an integer which is a multiple of p. But $f^{(p-1)}(0) = (n!)^p$ and since $p > n$ and is a prime number, $p \nmid (n!)^p$ so that *$f^{(p-1)}(0)$ is an integer not divisible by p.* Since $F(0) = f(0) + f^{(1)}(0) + \cdots + f^{(p-2)}(0) + f^{(p-1)}(0) + f^{(p)}(0) + \cdots + f^{(r)}(0)$, we conclude that *$F(0)$ is an integer not divisible by p.* Because $c_0 > 0$ and $p > c_0$ and because $p \nmid F(0)$ whereas $p \mid F(1)$, $p \mid F(2)$, $\ldots p \mid F(n)$, we can assert that *$c_0F(0) + c_1F(1) + \cdots + c_nF(n)$ is an integer and is not divisible by p.*

However, by (3), $c_0F(0) + c_1F(1) + \cdots + c_nF(n) = c_1\epsilon_1 + \cdots + c_n\epsilon_n$. What can we say about ϵ_i? Let us recall that

$$\epsilon_i = \frac{-e^{i(1-\theta_i)}(1 - i\theta_i)^p \ldots (n - i\theta_i)^p(i\theta_i)^{p-1}i}{(p-1)!},$$

where $0 < \theta_i < 1$. Thus

$$|\epsilon_i| \leq e^n \frac{n^p(n!)^p}{(p-1)!}.$$

As $p \to \infty$,

$$\frac{e^n n^p (n!)^p}{(p-1)!} \to 0,$$

(Prove!) whence we can find a prime number larger than both c_0 and n and large enough to force $|c_1\epsilon_1 + \cdots + c_n\epsilon_n| < 1$. But $c_1\epsilon_1 + \cdots + c_n\epsilon_n = c_0F(0) + \cdots + c_nF(n)$, *so must be an integer;* since it is smaller than 1 in size our only possible conclusion is that $c_1\epsilon_1 + \cdots + c_n\epsilon_n = 0$. Consequently, $c_0F(0) + \cdots + c_nF(n) = 0$; this however is sheer nonsense, since we know that $p \nmid (c_0F(0) + \cdots + c_nF(n))$, whereas $p \mid 0$. This contradiction, stemming from the assumption that e is algebraic, proves that e must be transcendental.