Assignment 3

Technion 236646

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August 10, 2022

Question 1 Let $G = (U, V, E)$ be a bipartite graph satisfying the following properties, for some constants d and $\alpha > 0$:

(i) $|U| = n + 1$ and $|V| = n$.

(ii) Every vertex in U has degree d.

(iii) If $S \subseteq U$ has size at most αn then there are at least $|S|$ vertices in V which are neighbors of exactly one vertex in S.

The *graph pigeonhole principle* corresponding to G has variables x_{uv} for all $(u, v) \in E$, and the following axioms:

- Pigeon axioms: For every $u \in U$, the axiom $\bigvee_{v\colon (u,v)\in E} x_{uv}$.
- Hole axioms: For every $u_1, u_2 \in U$ and $v \in V$ such that $(u_1, v), (u_2, v) \in E$ and $u_1 \neq u_2$, the following axiom: $\overline{x_{u_1v}} \vee \overline{x_{u_2v}}$.

(a) Show that the graph pigeonhole principle is unsatisfiable.

Answer Consider any assignment x. For each $u \in U$, let $f(u) \in V$ be some $v \in V$ such that $x_{uv} = 1$ (such v exists due to the pigeon axioms). Since $|U| > |V|$, the function f cannot be one-to-one. Suppose that $f(u_1) = f(u_2) = v$ for some $u_1 \neq u_2$. Then the hole axiom $\overline{x_{u_1v}} \vee \overline{x_{u_2v}}$ doesn't hold.

(b) For a clause C, let $w(C)$ be the minimal number of pigeon axioms which, together with *all* hole axioms, logically imply C. Show that $w(\perp) > \alpha n$. Hint: use Hall's theorem.

Answer Let $S \subseteq U$ contain at most αn vertices. Since every subset $T \subseteq S$ has at least $|T|$ neighbors in V, Hall's theorem shows that there is a matching $f\colon S \to V$. Consider the assignment in which the only variables set to 1 are $x_{uf(u)}$ for all $u \in S$. This assignment satisfies all pigeon axioms corresponding to vertices in S and all hole axioms, showing that these axioms do not imply $\perp$.

(c) Show that if $w(C) \leq \alpha n$ then C has width at least $w(C)$.

Answer Let $S \subseteq U$ be a minimal set of vertices whose pigeon axioms, together with all hole axioms, logically imply C. We will show that if $v \in V$ is the neighbor of precisely one vertex $u \in S$ then v must appear in C (that is, C must mention some variable $x_{u'v}$), hence C must have width at least $|S| = w(C)$.

Suppose, for the sake of contradiction, that v does not appear in C. Since S is minimal, there is a truth assignment x which satisfies all hole axioms and the pigeon axioms corresponding to $S \setminus \{u\}$, but falsifies C. Modify this assignment by setting $x_{uv} = 1$ and $x_{u'v} = 0$ for all $u' \neq u$ such that $(u', v) \in E$. By construction, $u' \notin S$ for all such u', and so the new assignment satisfies all hole axioms and the pigeon axioms corresponding to S. Since v does not appear in C, the clause C is still falsified, contrary to the assumption.

(d) Deduce that the width of refuting the graph pigeonhole principle is $\Omega(\alpha n)$.

Answer Consider any refutation of the principle. Explore the proof by starting at $\perp$ and repeatedly taking a child of maximal $w(\cdot)$ until reaching a clause C satisfying $w(C) \leq \alpha n$; such a clause must eventually be reached, assuming $n > 1/\alpha$, since $w(C) \leq 1$ for axioms C (if $n \leq 1/\alpha$, then the item trivially holds). Since $w(\perp) > \alpha n$, necessarily $w(C) \geq \alpha n/2$. Therefore C has width at least $\alpha n/2$.

(e) Conclude that the length required to refute the graph pigeonhole principle is $2^{\Omega(\alpha^2 n/d)}$.

Answer The number of variables is $N = d|U| = d(n+1)$. Section 5.5 shows that every refutation of the principle contains this many lines:

$$\exp \frac{\big(\Omega(\alpha n) - \max(d, 2)\big)^2}{N} = 2^{\Omega(\alpha^2 n/d)}.$$

(f) It is known that there exist constant d, α such that a random subgraph of $K_{n+1,n}$ (the complete bipartite graph on $(n+1)+n$ vertices) satisfying property (ii) also satisfies property (iii) with probability $1 - o(1)$.

The *pigeonhole principle* is the graph pigeonhole principle corresponding to $K_{n+1,n}$. Show that the length required to refute the pigeonhole principle is $2^{\Omega(n)}$.

Answer Let G be a subgraph of $K_{n+1,n}$ satisfying property (iii). Refuting the graph pigeonhole principle corresponding to G requires length $2^{\Omega(n)}$. A refutation of the pigeonhole principle can be converted to a refutation of the graph pigeonhole principle by substituting $x_{uv} = 0$ for all $(u, v) \notin G$. Therefore refuting the pigeonhole principle also requires length $2^{\Omega(n)}$.

In more detail, suppose that we are given a refutation of the pigeonhole principle. The substitution $x_{uv} = 0$ for all $(u, v) \notin G$ has the following effect. Some lines become satisfied (if they contain a literal x_{uv} for some $(u, v) \notin G$), and can be removed from the proof. Pigeon axioms become pigeon axioms. Hole axioms either stay the same or become satisfied, and so can be removed from the proof. Weakening steps remain weakening steps. The cut rule $C \lor x, D \lor \bar{x} \vdash C \lor D$ either remains an instance of the cut rule (if x was not removed), or becomes $C \vdash C'$ for some weakening C' of C (if x was removed). Therefore the substitution results in a refutation of the graph pigeonhole principle corresponding to G which contains at most as many lines as the original proof.

Question 2 Tarsi showed that if S is a minimally unsatisfiable set of clauses (meaning that S is unsatisfiable, but any proper subset is satisfiable) then S mentions fewer than $|S|$ variables.

(a) Suppose that S is an unsatisfiable set of clauses. Show that S can be refuted in Resolution in length $O(2^{|S|})$.

Answer Let $T \subseteq S$ be a minimally unsatisfiable subset of T. According to Tarsi's result, T contains fewer than $|T|$ variables, and so can be refuted in length $O(2^{|T|}) = O(2^{|S|})$.

(b) Suppose that S logically implies a clause C. Shows that C can be derived from S in Resolution (including the weakening rule) in length $O(2^{|S|})$.

Answer Let $C = \ell_1 \lor \cdots \lor \ell_m$. Let S' be the result of substituting $\ell_1 = \cdots = \ell_m = 0$ in S', that is, removing all clauses containing any $\overline{\ell_i}$, and removing any ℓ_j from the remaining clauses. We can refute S' in length $N = O(2^{|S|})$. Now apply the same proof, replacing each axiom of S' with the corresponding axioms of S. Induction shows that for every line D in the original proof, the new proof contains a line $D \lor D'$, where D' is some strengthening of C. In particular, the final line is some strengthening of C. We can derive C using one application of the weakening rule, obtaining a proof of length $N + 1 = O(2^{|S|})$.

(c) Semantic d-ary Resolution is the following proof system. Each line is a clause, which is either an axiom, or logically follows from up to d earlier lines. (Resolution is essentially the case $d = 2$, if we allow the "weakening cut" rule $C \lor x, D \lor \bar{x} \vdash C \lor D \lor E$.)

Show that if a CNF can be refuted in Semantic d-ary Resolution using ℓ lines, then it can also be refuted in Resolution using $O(2^d \ell)$ lines.

Answer We can replace every step using a Resolution derivation of length $O(2^d)$.

(d) Show that for every constant d, we can check the validity of a Semantic d-ary Resolution proof in time $2^{O(d)} n^{O(1)}$.

Answer Let $C_1, \ldots, C_d \vdash C$ be an inference in the system, where $C_1, \ldots, C_d, C$ are clauses. This inference is valid iff $C'_{i_1}, \ldots, C'_{i_D} \vdash \bot$, where C'_i is obtained from C_i by removing all literals in C, and removing the entire clause if it contains the negation of some literal in C. If S is a minimally unsatisfiable subset of $C'_{i_1}, \ldots, C'_{i_D}$ then S mentions fewer than $|S| \leq d$ variables, and so $C'_{i_1}, \ldots, C'_{i_D} \vdash \bot$ iff there is an unsatisfiable subset of $C'_{i_1}, \ldots, C'_{i_D}$ which mentions at most d variables. We can check the latter condition in time $O(4^d)$.

Question 3 Consider a random walk process on the line $\{0, \ldots, n\}$ satisfying the following two properties:

1. If at time t the walk is at n, then at time $t+1$ it is at $n-1$.
2. If at time t the walk is at $0 < d < n$, then at time $t+1$ it is either at $d-1$ or at $d+1$; and furthermore, it is at $d-1$ with probability at least $1/2$.

It is known that such a process hits 0 after $O(n^2)$ steps, in expectation. Use this to analyze the performance of Schöning's algorithm on 2CNFs.

Answer Suppose that the 2CNF has some satisfying assignment x^*. The distance between the current assignment x and x^* behaves like the random walk process described in the question. Consequently, Schöning's algorithm finds x^* (or some other satisfying assignment) in expected time $O(n^2)$.