Assignment 3

Technion 236646

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Question 1 Let $G = (U, V, E)$ be a bipartite graph satisfying the following properties, for some constants d and $\alpha > 0$:

(i) $|U| = n + 1$ and $|V| = n$.

(ii) Every vertex in U has degree d.

(iii) If $S \subseteq U$ has size at most αn then there are at least $|S|$ vertices in V which are neighbors of exactly one vertex in S.

The *graph pigeonhole principle* corresponding to G has variables x_{uv} for all $(u, v) \in E$, and the following axioms:

- Pigeon axioms: For every $u \in U$, the axiom $\bigvee_{v\colon (u,v)\in E} x_{uv}$.
- Hole axioms: For every $u_1, u_2 \in U$ and $v \in V$ such that $(u_1, v), (u_2, v) \in E$ and $u_1 \neq u_2$, the following axiom: $\overline{x_{u_1 v}} \vee \overline{x_{u_2 v}}$.

(a) Show that the graph pigeonhole principle is unsatisfiable.

(b) For a clause C, let $w(C)$ be the minimal number of pigeon axioms which, together with *all* hole axioms, logically imply C. Show that $w(\bot) > \alpha n$. Hint: use Hall's theorem.

(c) Show that if $w(C) \leq \alpha n$ then C has width at least $w(C)$.

(d) Deduce that the width of refuting the graph pigeonhole principle is $\Omega(\alpha n)$.

(e) Conclude that the length required to refute the graph pigeonhole principle is $2^{\Omega(\alpha^2 n/d)}$.

(f) It is known that there exist constant d, α such that a random subgraph of $K_{n+1,n}$ (the complete bipartite graph on $(n + 1) + n$ vertices) satisfying property (ii) also satisfies property (iii) with probability $1 - o(1)$.

The *pigeonhole principle* is the graph pigeonhole principle corresponding to $K_{n+1,n}$. Show that the length required to refute the pigeonhole principle is $2^{\Omega(n)}$.

Question 2 Tarsi showed that if S is a minimally unsatisfiable set of clauses (meaning that S is unsatisfiable, but any proper subset is satisfiable) then S mentions fewer than $|S|$ variables.

(a) Suppose that S is an unsatisfiable set of clauses. Show that S can be refuted in Resolution in length $O(2^{|S|})$.

(b) Suppose that S logically implies a clause C. Shows that C can be derived from S in Resolution (including the weakening rule) in length $O(2^{|S|})$.

(c) Semantic d-ary Resolution is the following proof system. Each line is a clause, which is either an axiom, or logically follows from up to d earlier lines. (Resolution is essentially the case $d = 2$, if we allow the "weakening cut" rule $C \vee x, D \vee \bar{x} \vdash C \vee D \vee E$.)

Show that if a CNF can be refuted in Semantic d-ary Resolution using ℓ lines, then it can also be refuted in Resolution using $O(2^d \ell)$ lines.

(d) Show that for every constant d, we can check the validity of a Semantic d-ary Resolution proof in time $2^{O(d)} n^{O(1)}$.

Question 3 Consider a random walk process on the line $\{0, \ldots, n\}$ satisfying the following two properties:

1. If at time t the walk is at n, then at time $t + 1$ it is at $n - 1$.
2. If at time t the walk is at $0 < d < n$, then at time $t + 1$ it is either at $d - 1$ or at $d + 1$; and furthermore, it is at $d - 1$ with probability at least $1/2$.

It is known that such a process hits 0 after $O(n^2)$ steps, in expectation. Use this to analyze the performance of Schöning's algorithm on 2CNFs.