Assignment 1

Technion 236646

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April 19, 2022

For the sake of this assignment, assume that P≠NP.

Question 1 Suppose that Γ is an infinite set of predicates whose only polymorphisms are projections and (possibly) anti-projections.

(a) Show that Γ has a finite subset with this property.

(b) Bonus: Define a version of CSP(Γ) which makes sense for (some) infinite Γ, and prove a dichotomy theorem for it.

Question 2 For $0 \leq i \leq n$, let $\alpha_{n,i}$ be the following n-ary predicate:

$$\alpha_{n,i} = \{(x_1, \ldots, x_n) \in \{0,1\}^n : x_1 + \cdots + x_n = i\}.$$

(a) Let $A = \{\alpha_{n,i} : n \in \mathbb{N}, 0 \leq i \leq n\}$. Determine for which finite sets $\Gamma \subseteq A$ the problem CSP(Γ) is in P, and for which the problem CSP(Γ) is NP-complete.

(b) Now suppose that $\Gamma \subseteq A$ is infinite. We extend the definition of CSP(Γ) to infinite Γ in the natural way: a constraint $\alpha_{n,i}(x_{i_1}, \ldots, x_{i_n})$ is encoded as a tuple $(n, i, i_1, \ldots, i_n)$, and an instance is a Yes instance if there is a satisfying assignment; we are promised that all constraints in the input instance belong to Γ.

Show that CSP(Γ) is either NP-complete or in P.

Question 3 Suppose that Γ is a finite set of predicates which is closed under negating individual inputs: if $R \in \Gamma$ is a k-ary predicate, then for each $i \in \{1, \ldots, k\}$, the predicate $R^{(i)}(x_1, \ldots, x_k) = R(x_1, \ldots, x_{i-1}, \neg x_i, x_{i+1}, \ldots, x_k)$ is also in Γ. Examples of CSPs satisfying this condition include kSAT, kNAE-SAT, and kXOR-SAT.

(a) Show that CSP(Γ) is in P iff either ternary majority or ternary XOR is a polymorphism of Γ.

(b) Show that there is no single polymorphism f such that CSP(Γ) is in P iff f is a polymorphism of Γ.

Question 4 In this question, we consider predicates and polymorphisms over a finite alphabet Σ. A *unital* function is a function depending on a single coordinate.

Let Γ be a finite set of relations. Suppose that Γ has no constant polymorphisms, and that it has some non-unital polymorphism. Show that Γ has a non-unital polymorphism depending on at most $|\Sigma| + 1$ coordinates.